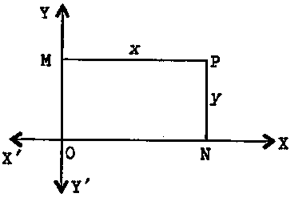
10 Straight Lines

Lesson at a Glance

1. **Definition : coordinates of a point**

   If P(x, y) are the coordinates of a point, then



   (i) perpendicular distance of the point P(x, y) from y-axis is x and is called **abscissa** of the point P.

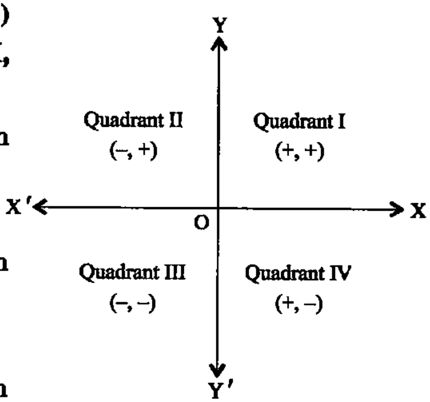
   (ii) perpendicular distance of the point P(x, y) from x-axis is y and is called **ordinate** of the point P.

2. x-axis and y-axis taken together are called coordinate axes.

   Equation of x-axis is $y = 0$.

   Equation of y-axis is $x = 0$.

3. **Rule for a point (x, y) to lie in quadrant I, II, III or IV**



   (i) For point (x, y) in first quadrant,

   $x > 0, y > 0.$

   (ii) For point (x, y) in second quadrant,

   $x < 0, y > 0.$

   (iii) For point (x, y) in third quadrant,

   $x < 0, y < 0.$

   (iv) For point (x, y) in fourth quadrant, $x > 0, y < 0.$

4. Distance formula

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points.

Then distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(\text{Difference of abscissa})^2 + (\text{difference of ordinates})^2}$

5. Section formula

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two given points. Then

(*i*) coordinates of the point R which divides the join of points P and Q **internally** (*i.e.*, point R lies within PQ) in the ratio

$m_1 : m_2 \left(i.e., \frac{PR}{QR} = \frac{m_1}{m_2}\right)$ are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right).$$

(*ii*) coordinates of the point R which divides the join of points P and Q **externally** (*i.e.*, point R lies outside the segment PQ *i.e.*, either point R lies to the right of Q or to the left of P)

are $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$.

6. To find the ratio in which the point R divides PQ, it is convenient to take the ratio as $k : 1$.

7. Mid-point formula

The coordinates of the mid-point of the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

8. Centroid of a triangle

The point of concurrence of the medians of a triangle (median is a line joining a vertex of the triangle to the mid-point of the opposite side) is called **centroid** of the triangle. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the three vertices of a given triangle ABC. Then, the centroid of ΔABC is given by

$$G \leftrightarrow \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

9. Area of a triangle

If A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) are the vertices of a triangle ABC, then area of ΔABC is $\frac{1}{2}\,|\,x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\,|$.

10. Condition of collinearity of three points

Three points A, B, C are collinear if area of ΔABC is zero. *i.e.*, if $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$.

11. Slope of a line.

Definition 1. If a line makes an angle θ with positive direction of x-axis, then its slope $m = \tan\theta$.

Cor I. If a line is parallel to x-axis, its slope is 0.

Cor II. If a line is parallel to y-axis, its slope is $\tan 90° = \infty$ and hence undefined.

12. Condition for two lines having slopes m_1 and m_2 to be parallel is $m_1 = m_2$.

13. Condition for two lines having slopes m_1 and m_2 to be perpendicular is $m_1 m_2 = -1$.

i.e., $m_2 = \frac{-1}{m_1}$ (Negative reciprocal.)

14. Slope of a line

Definition 2. Slope of the line joining two points A(x_1, y_1) and B(x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{y_1 - y_2}{x_1 - x_2}$$

i.e., Slope of a line = $\frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$.

15. Condition of collinearity of three points

(Better condition than given in result 10.)

Three points A, B, C will be **collinear** if and only if **slope of AB = slope of BC.**

16. Slope of a line

Definition 3. Slope of the line $ax + by + c = 0$ is

$$-\frac{a}{b} \quad i.e., \; -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}.$$

17. Locus and its equation

Locus of a point is the path traced by the moving point under a given condition(s).

When a point moves under some condition, then the algebraic relation between the co-ordinates (x, y) of this moving point is called the **equation of the locus** of that point.

18. Equations of lines parallel to x-axis and parallel to y-axis.

(*a*) Equation of the line parallel to y-axis and at distance a from it is $x = a$.

If $a > 0$, the line is to the right of y-axis.

If $a < 0$, the line is to the left of y-axis.

(*b*) Equation of the line parallel to x-axis and at a distance b from it is $y = b$.

If $b > 0$, the line is above the x-axis.

If $b < 0$, the line is below the x-axis.

19. Equation of a line passing through the origin is $y = mx$ where m is the slope of the line.

20. The point-slope form (or one-point form) of the equation of a line.

Equation of straight line passing through a point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

Special cases:

(*i*) Equation of the line through (x_1, y_1) and parallel to x-axis is $y = y_1$.

(*ii*) Equation of the line through (x_1, y_1) and parallel to y-axis is $x = x_1$.

21. Two point form of the equation of a line

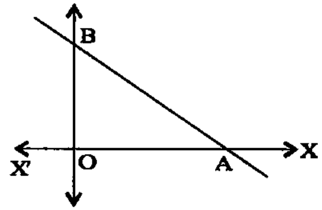
Equation of the straight line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Note. By result 14, $\frac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line.

22. Intercepts

If a straight line cuts the axes of x and y in the points A and B respectively, then length OA is called the intercept on x-axis and length OB is called intercept on y-axis.



Portion of the line intercepted between the coordinate axes = AB

$$= \sqrt{OA^2 + OB^2}.$$

23. Slope intercept form of the equation of a line.

Equation of a line having slope m and making an intercept c on y-axis is $y = mx + c$.

24. Intercept form of the equation of a line

Equation of a line making an intercept a on x-axis and an intercept b on y-axis is $\frac{x}{a} + \frac{y}{b} = 1$.

25. Perpendicular (or normal) form of the equation of a line.

Equation of a line on which the length of the perpendicular from the origin is p and this perpendicular segment makes an angle ω with the x-axis is $x \cos \omega + y \sin \omega = p$.

26. Angle between two lines.

If θ is the acute angle between the two lines having slopes m_1 and m_2 respectively, then $\tan \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

27. Equation of a line parallel to a given line.

The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + k = 0$ where k is a constant.

Rule. Keep the terms of x and y unaltered and change the constant term to k.

28. Equation of a line perpendicular to a given line.

The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + k = 0$ where k is a constant.

Rule. Interchange the coefficients of x and y, change the sign of one of them and change the constant term to k.

29. Test for two lines to be intersecting, parallel or coincident.

Let the two lines be $a_1x + b_1y + c_1 = 0$

and $a_2x + b_2y + c_2 = 0$.

Then (*i*) the lines are **intersecting** if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(*ii*) **parallel** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(*iii*) **coincident** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

30. Method to find point of intersection of two lines.

Solve the equations of the two lines for x and y.

31. Concurrent lines

Three lines are said to be **concurrent** if they pass through a common point *i.e.,* if the point of intersection of any two of them lies on the third.

32. Orthocentre of a triangle

The point of intersection of the altitudes of a triangle is called **orthocentre** of the triangle.

33. Perpendicular distance of a point from a line

The length of the perpendicular from a given point (x_1, y_1) on the line $ax + by + c = 0$ is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

34. Distance between two parallel lines

Distance between two parallel lines

$$ax + by + c = 0$$

and $ax + by + c' = 0$ is $\frac{|c - c'|}{\sqrt{a^2 + b^2}}$.

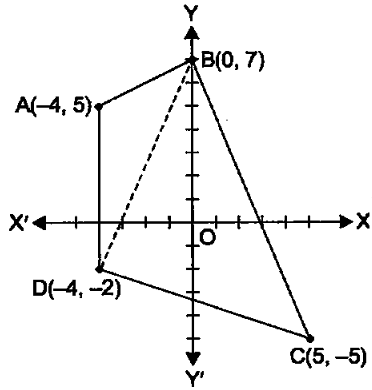
TEXTBOOK QUESTIONS SOLVED

EXERCISE 10.1 (*Page No.: 211–212*)

1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, – 2). Also, find its area.

Sol. We know that area of quadrilateral ABCD

= Area of triangle ABD + Area of triangle BCD



[Using the formula for area of triangle

$$= \frac{1}{2}\,|\,x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\,|$$

for both triangles ABD and BCD]

Area of quadrilateral ABCD

$$= \frac{1}{2}\,|-4(7+2) + 0(-2-5) - 4(5-7)\,|$$

$$+ \frac{1}{2}\,|\,0(-5+2) + 5(-2-7) - 4(7+5)\,|$$

$$= \frac{1}{2}\,|-36+8\,| + \frac{1}{2}\,|-45-48\,|$$

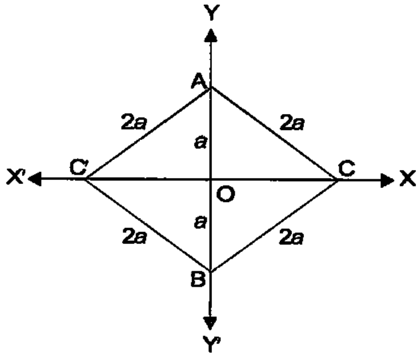
$$= \frac{1}{2}(28) + \frac{1}{2}(93) = 14 + 46.5$$

$= 60.5$ square units.

2. **The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.**

Sol. Let the base AB of an equilateral triangle ABC of side $2a$ lie along the y-axis such that origin O is the mid-point of AB (given). Therefore, OA = OB = a and co-ordinates of A are $(0, a)$ and those of B are $(0, -a)$. ($\because$ on y-axis $x = 0$)

Since side AB is y-axis (given) and O, the mid-point of AB, therefore OC is the right bisector of AB ($\because$ Δ ABC is an equilateral) and is x-axis.



Now $\quad OC = \sqrt{AC^2 - OA^2}$

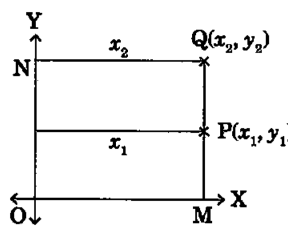
(By Pythagoras Theorem)

$= \sqrt{(2a)^2 - a^2} = \sqrt{3a^2} = \sqrt{3}\, a$

Similarly $OC' = \sqrt{3}\, a$.

$\therefore$ The co-ordinates of the third vertex C are $(\sqrt{3}\, a, 0)$ and those of C' are $(-\sqrt{3}\, a, 0)$.

3. Find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ when: (*i*) PQ is parallel to the *y*-axis, (*ii*) PQ is parallel to the *x*-axis.



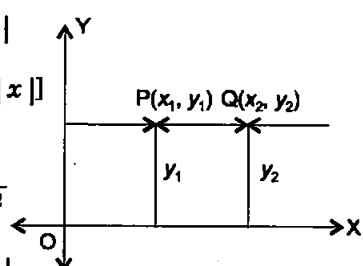
Sol. (*i*) When PQ is parallel to the *y*-axis, $x_1 = x_2$.

$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$

$[\because \quad \sqrt{x^2} = |x|]$

(*ii*) When PQ is parallel to the *x*-axis, $y_1 = y_2$.



$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$.

4. Find a point on the *x*-axis, which is equidistant from the points (7, 6) and (3, 4).

Sol. Let the point P(*x*, 0) on the *x*-axis ($\because$ on *x*-axis, $y = 0$) be equidistant from the points A(7, 6) and B(3, 4).

Then $PA = PB$

$$\Rightarrow \quad \sqrt{(x-7)^2+(0-6)^2} = \sqrt{(x-3)^2+(0-4)^2}$$

Squaring, $x^2 - 14x + 49 + 36 = x^2 - 6x + 9 + 16$

$$\Rightarrow \quad -14x + 85 = -6x + 25$$

$$\Rightarrow \quad -8x = -60 \quad \therefore \quad x = \frac{60}{8} = \frac{15}{2}$$

$\therefore$ The required point on x-axis is $\left(\frac{15}{2}, 0\right)$.

5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P(0, – 4) and B(8, 0).

Sol. Let M be the mid-point of PB, then by mid-point formula;

$$M = \left(\frac{0+8}{2}, \frac{-4+0}{2}\right) = (4, -2).$$

The origin O has coordinates (0, 0).

$$\therefore \quad \text{Slope of OM} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-0}{4-0} = \frac{-2}{4} = -\frac{1}{2}.$$

6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (– 1, – 1) are the vertices of a right-angled triangle.

Sol. Let the three given points be A(4, 4), B(3, 5) and C(– 1, – 1).

$$m_1 = \text{slope of AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-4}{3-4} = \frac{1}{-1} = -1$$

$$m_2 = \text{slope of BC} = \frac{-1-5}{-1-3} = \frac{-6}{-4} = \frac{3}{2}$$

$$m_3 = \text{slope of AC} = \frac{-1-4}{-1-4} = \frac{-5}{-5} = 1$$

Here $m_1 . m_3 = (-1)(1) = -1$.

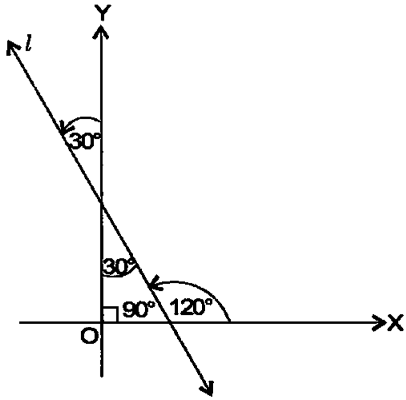
$\therefore$ Lines AB and AC are perpendicular to each other.

$\therefore$ Triangle ABC is a right-angled triangle with angle A = 90°.

$\therefore$ The given points are the vertices of a right-angled triangle.

7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.

Sol. The line makes an angle of 30° with the positive direction of y-axis, *i.e.*, with OY, measured anticlockwise.



$\therefore$ The line makes an angle 90° + 30° = 120° with the positive direction of x-axis.

$\therefore$ Slope of the line

$$= \tan 120°$$
$$= \tan (180° - 60°)$$
$$= -\tan 60° = -\sqrt{3}.$$

8. Find the value of x for which the points (x, – 1), (2, 1) and (4, 5) are collinear.

Sol. Given: The points A(x, – 1), B(2, 1) and C(4, 5) are collinear.

$\Rightarrow$ Slope of AB = Slope of BC.

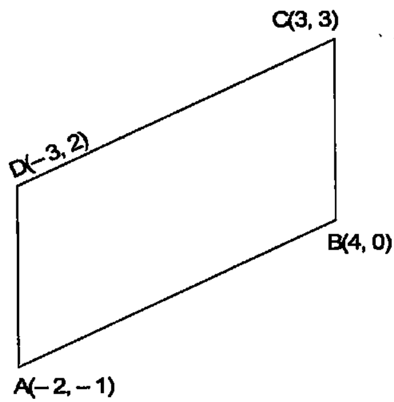
$$\Rightarrow \quad \frac{1-(-1)}{2-x} = \frac{5-1}{4-2}$$

$$\Rightarrow \quad \frac{2}{2-x} = 2 \qquad \Rightarrow \quad 2 = 4 - 2x$$

$$\Rightarrow \quad 2x = 2 \qquad \therefore \quad x = 1.$$

9. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (– 3, 2) are the vertices of a parallelogram.

Sol. The given points are A(– 2, – 1), B(4, 0), C(3, 3) and D(–3, 2).



Slope of AB $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0-(-1)}{4-(-2)} = \dfrac{1}{6}$.

Slope of DC $= \dfrac{3-2}{3-(-3)} = \dfrac{1}{6}$.

$\Rightarrow$ Slope of AB = Slope of DC.

$\Rightarrow$ AB $\parallel$ DC ...(*i*)

Slope of BC $= \dfrac{3-0}{3-4} = \dfrac{3}{-1} = -3$

Slope of AD $= \dfrac{2-(-1)}{-3-(-2)} = \dfrac{2+1}{-3+2} = \dfrac{3}{-1} = -3$

$\Rightarrow$ Slope of BC = Slope of AD

$\Rightarrow$ BC $\parallel$ AD ...(*ii*)

Since the two pairs of opposite sides of the quadrilateral ABCD are parallel, therefore, ABCD is a parallelogram.

10. Find the angle between the x-axis and the line joining the points (3, – 1) and (4, – 2).

Sol. Slope of the line joining the points A(3, – 1) and B(4, – 2) is

$$\frac{-2-(-1)}{4-3} = \frac{-2+1}{1} = -1.$$

If the line AB makes an angle θ with the positive direction of x-axis, then its slope is tan θ.

$\therefore$ $\tan\theta = -1 = -\tan 45° = \tan(180° - 45°) = \tan 135°$

$\Rightarrow$ $\theta = 135°$.

11. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.

Sol. Let the slopes of two lines be m and $2m$. If θ is the angle between them, then

$$\tan\theta = \frac{1}{3}$$

Using $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \quad \left|\frac{2m - m}{1 + 2m.m}\right| = \frac{1}{3} \quad \Rightarrow \quad \left|\frac{m}{1 + 2m^2}\right| = \frac{1}{3}$$

$$\Rightarrow \quad \frac{m}{1 + 2m^2} = \pm\frac{1}{3} \qquad (\because |x| = a \ (a \geq 0) \Rightarrow x \pm a)$$

When $\frac{m}{1 + 2m^2} = \frac{1}{3}$, we have

$$3m = 1 + 2m^2.$$

$$\Rightarrow \quad 2m^2 - 3m + 1 = 0 \qquad \Rightarrow \quad 2m^2 - 2m - m + 1 = 0$$

$$\Rightarrow \quad 2m(m - 1) - (m - 1) = 0 \qquad \Rightarrow \quad (m - 1)(2m - 1) = 0$$

$$\Rightarrow \quad m = 1, \frac{1}{2}$$

When $\frac{m}{1 + 2m^2} = -\frac{1}{3}$, we have

$$-3m = 1 + 2m^2$$

$$\Rightarrow \quad 2m^2 + 3m + 1 = 0 \qquad \Rightarrow \quad (m + 1)(2m + 1) = 0$$

$$\Rightarrow \quad m = -1, -\frac{1}{2}$$

Combining, $m = 1, \frac{1}{2}, -1, -\frac{1}{2}$

Hence, the slopes of the two lines are: (m and $2m$) 1 and 2 or $\frac{1}{2}$ and 1 or – 1 and – 2 or $-\frac{1}{2}$ and – 1.

12. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that $k - y_1 = m(h - x_1)$.

Sol. Slope of the line passing through (x_1, y_1) and (h, k) is

$$\frac{k - y_1}{h - x_1} = m \qquad \text{(given)}$$

$\Rightarrow$ Cross multiplying $k - y_1 = m(h - x_1)$.

13. If three points $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

Sol. Since points $P(h, 0)$, $Q(a, b)$ and $R(0, k)$ lie on a line, they are collinear.

$\therefore$ Slope of PQ = Slope of QR

$$\Rightarrow \frac{b - 0}{a - h} = \frac{k - b}{0 - a} \qquad \Rightarrow \frac{b}{a - h} = \frac{k - b}{-a}$$

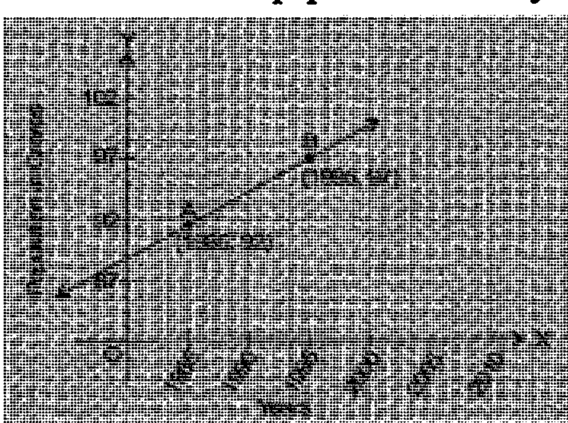
$$\Rightarrow \quad -ab = (a - h)(k - b) \Rightarrow -ab = ak - ab - hk + hb$$

$$\Rightarrow \quad 0 = ak - hk + hb \Rightarrow \quad hk = ak + hb$$

Dividing by hk, we have

$$1 = \frac{a}{h} + \frac{b}{k} \quad \text{or} \quad \frac{a}{h} + \frac{b}{k} = 1.$$

14. Consider the following population and year graph (see figure), find the slope of the line AB and using it, find what will be the population in the year 2010?



Sol. Slope of line AB $= \frac{97 - 92}{1995 - 1985} = \frac{5}{10} = \frac{1}{2}$.

Let the population in 2010 be p crores, then the point $C(2010, p)$ lies on AB.

$\therefore$ Slope of AB = Slope of BC.

$$\Rightarrow \quad \frac{1}{2} = \frac{p-97}{2010-1995} \quad \Rightarrow \quad \frac{1}{2} = \frac{p-97}{15}$$

$$\Rightarrow \quad \frac{15}{2} = p - 97 \quad \Rightarrow \quad 7.5 + 97 = p$$

$$\Rightarrow \quad p = 104.5$$

Hence, the population in the year 2010 is 104.5 crores.

EXERCISE 10.2 *(Page No.: 219–220)*

In Exercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for the x- and y-axis.

Sol. Let P(x, y) be any point on x-axis. Since the ordinate of every point on x-axis is zero, we have $y = 0$.

Hence, the equation of x-axis is $y = 0$.

Let P(x, y) be any point on y-axis. Since the abscissa of every point on y-axis is zero, we have $x = 0$.

Hence, the equation of y-axis is $x = 0$.

2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.

Sol. Here $x_1 = -4$, $y_1 = 3$ and $m = \frac{1}{2}$.

Using point-slope form, equation of line is $y - y_1 = m(x - x_1)$

$$\text{or} \quad y - 3 = \frac{1}{2}(x + 4)$$

$$\text{or} \quad 2y - 6 = x + 4$$

$$\text{or} \quad x - 2y + 10 = 0.$$

3. Passing through (0, 0) with slope m.

Sol. Here $x_1 = 0$, $y_1 = 0$.

Using point-slope form, equation of line is $y - 0 = m(x - 0)$

$$\text{or} \quad y = mx.$$

4. Passing through $(2, 2\sqrt{3})$ and inclined with the x-axis at an angle of 75°.

Sol. Here m, the slope of line = tan 75° = tan (45° + 30°)

$$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Also, $x_1 = 2, y_1 = 2\sqrt{3}$.

Using point-slope form, equation of line is

$$y - 2\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1}(x - 2)$$

cross-multiplying,

or $(\sqrt{3} - 1)y - 6 + 2\sqrt{3} = (\sqrt{3} + 1)x - 2\sqrt{3} - 2$

or $4\sqrt{3} - 4 = (\sqrt{3} + 1)x - (\sqrt{3} - 1)y$

or $(\sqrt{3} + 1)x - (\sqrt{3} - 1)y = 4(\sqrt{3} - 1)$.

5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope – 2.

Sol. Here, the line passes through the point (– 3, 0) (3 units to the left of origin on x-axis $\Rightarrow x = -3$ and $y = 0$) and has slope – 2.

$\therefore$ Equation of line is $y - 0 = -2(x - (-3))$

or $y = -2(x + 3)$ or $y = -2x - 6$

or $2x + y + 6 = 0$.

6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

Sol. Here the line passes through the point (0, 2) (2 units above the origin on y-axis $\Rightarrow x = 0$ and $y = 2$) and has slope

$$= \tan 30° = \frac{1}{\sqrt{3}}.$$

$\therefore$ Equation of line is $y - 2 = \frac{1}{\sqrt{3}}(x - 0)$

or $\sqrt{3}y - 2\sqrt{3} = x$ or $x - \sqrt{3}y + 2\sqrt{3} = 0$.

7. Passing through the points (– 1, 1) and (2, – 4).

Sol. Here, $x_1 = -1, y_1 = 1, x_2 = 2, y_2 = -4$.

Using two-point form, equation of line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $y - 1 = \frac{-4-1}{2-(-1)}(x - (-1))$ or $y - 1 = \frac{-5}{3}(x + 1)$

or $3y - 3 = -5x - 5$ or $5x + 3y + 2 = 0$.

8. Perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x-axis is 30°.

Sol. Here, $p = 5$ and $\omega = 30°$

Using normal form, equation of line is

$$x \cos \omega + y \sin \omega = p$$

or $x \cos 30° + y \sin 30° = 5$ or $x.\frac{\sqrt{3}}{2} + y.\frac{1}{2} = 5$

Multiplying by 2, $\sqrt{3}\,x + y = 10$.

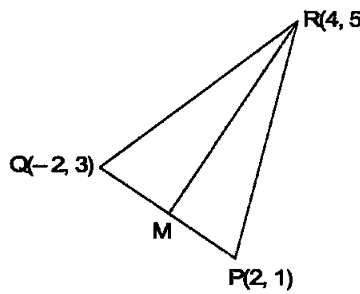
9. The vertices of ΔPQR are P(2, 1), Q(– 2, 3) and R(4, 5). Find equation of the median through the vertex R.

Sol. Let M be the mid-point of side PQ, then

$$M = \left(\frac{2-2}{2}, \frac{1+3}{2}\right) = (0, 2)$$

∴ PM is a median of ΔPQR.

Using two-point form, equation of median RM is



$$y - 5 = \frac{2-5}{0-4}(x - 4)$$

(Here $x_1 = 4, y_1 = 5, x_2 = 0, y_2 = 2$)

or $y - 5 = \frac{3}{4}(x - 4)$ or $4y - 20 = 3x - 12$

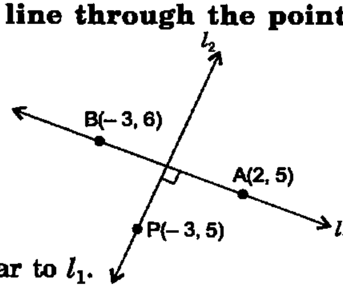
or $3x - 4y + 8 = 0$.

10. Find the equation of the line passing through (– 3, 5) and perpendicular to the line through the points (2, 5) and (– 3, 6).

Sol. Let l_1 be the line through the points A(2, 5) and B(– 3, 6).

Slope of $l_1 = \frac{6-5}{-3-2} = -\frac{1}{5}$.

Required line l_2 is perpendicular to l_1.



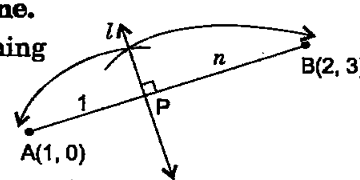
$\therefore$ Slope of $l_2 = 5$ [– ve reciprocal]

Equation of l_2 passing through P (– 3, 5) and having slope 5 is

$$y - 5 = 5(x - (-3)) \qquad \text{[Point-slope form]}$$

or $\quad y - 5 = 5x + 15 \quad$ or $\quad 5x - y + 20 = 0.$

11. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1 : *n*. Find the equation of the line.



Sol. Slope of the line segment joining A(1, 0) and B(2, 3) is $\dfrac{3-0}{2-1} = 3.$

Required line l is perpendicular to AB.

$\therefore$ Slope of $l = -\dfrac{1}{3}$ [– ve reciprocal]

Also, l divides AB internally in the ratio 1 : n at P. $\therefore$ By section formula,

$$\therefore \quad P = \left(\frac{1\times 2 + n\times 1}{1+n}, \frac{1\times 3 + n\times 0}{1+n}\right) = \left(\frac{2+n}{1+n}, \frac{3}{1+n}\right)$$

Using point-slope form, equation of line l is

$$y - \frac{3}{1+n} = -\frac{1}{3}\left(x - \frac{2+n}{1+n}\right)$$

cross-multiplying,

or $$3y - \frac{9}{1+n} = -x + \frac{2+n}{1+n}$$

or $\quad 3(1 + n)y - 9 = -(1 + n)x + 2 + n$

or $\quad (1 + n)x + 3(1 + n)y = n + 11.$

12. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).

Sol. Let the equal intercepts be a, a.

Then equation of line is

$$\frac{x}{a} + \frac{y}{a} = 1 \qquad \text{[Intercept form]}$$

or $\quad x + y = a \qquad$ ...(*i*)

Since it passes through the point (2, 3), we have

$$2 + 3 = a \quad \text{or} \quad a = 5$$

Putting $a = 5$ in (*i*), the equation of line is $x + y = 5$.

13. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

Sol. Let the intercept on x-axis be a, then the intercept on y-axis is $b = 9 - a$.

[$\because$ $a + b = 9$ (given)]

Therefore, equation of line in intercept form is

$$\frac{x}{a} + \frac{y}{9-a} = 1 \quad \text{...(i)} \qquad \left|\frac{x}{a}+\frac{y}{b}=1\right|$$

Since, it passes through the point (2, 2), we have

$$\frac{2}{a} + \frac{2}{9-a} = 1 \Rightarrow \frac{2(9-a)+2a}{a(9-a)}=1$$

cross-multiplying, we get

$$2(9 - a) + 2a = a(9 - a)$$

or $\quad 18 - 2a + 2a = 9a - a^2 \quad$ or $\quad a^2 - 9a + 18 = 0$

or $\quad (a - 3)(a - 6) = 0$

$\therefore \quad a = 3, 6$

When $a = 3$, from (*i*)

$$\frac{x}{3} + \frac{y}{6} = 1 \Rightarrow \frac{2x+y}{6}=1$$

or $\quad 2x + y = 6$ or $2x + y - 6 = 0$

When $a = 6$, from (*i*)

$$\frac{x}{6} + \frac{y}{3} = 1 \Rightarrow \frac{x+2y}{6}=1$$

or $\quad x + 2y = 6$ or $x + 2y - 6 = 0$

Hence, the equations of the line are

$$2x + y - 6 = 0, \; x + 2y - 6 = 0.$$

14. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

Sol. The line passes through the point (0, 2).

$$\text{Its slope} = \tan\frac{2\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$$

$\therefore$ Its equation is $y - 2 = -\sqrt{3}(x - 0)$ | $\because$ $y - y_1 = m(x - x_1)$

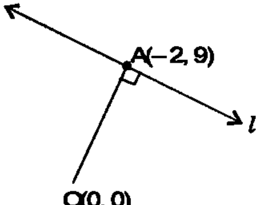
or $\qquad y = -\sqrt{3}\,x + 2 \qquad$ ...(*i*)

Any line parallel to (*i*) also has slope $-\sqrt{3}$. If it cuts the *y*-axis at a distance 2 below the origin, *i.e.*, if it passes through the point (0, – 2), then its equation is

$$y + 2 = -\sqrt{3}(x - 0) \quad \text{or } y + 2 = -\sqrt{3}\,x \text{ or } \quad \sqrt{3}\,x + y + 2 = 0$$

Hence, required equations of the lines are $\sqrt{3}\,x + y - 2 = 0$ and $\sqrt{3}\,x + y + 2 = 0$.

15. The perpendicular from the origin to a line meets it at the point (– 2, 9), find the equation of the line.



Sol. Slope of OA $= \dfrac{9 - 0}{-2 - 0} = -\dfrac{9}{2}$

Since OA $\perp$ *l*, slope of $l = \dfrac{2}{9}$. [– ve reciprocal]

Using point-slope form, equation of line *l* is

$$y - 9 = \frac{2}{9}(x - (-2))$$

cross-multiplying, $9(y - 9) = 2(x + 2)$

or $\qquad 9y - 81 = 2x + 4 \qquad$ or $\quad 2x - 9y + 85 = 0.$

16. The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L = 125.134 when C = 110, express L in terms of C.

Sol. Assuming C along *x*-axis and L along *y*-axis, we have two points (20, 124.942) and (110, 125.134) in XY-plane. By two-point form, the point (C, L) satisfies the equation

$$L - 124.942 = \frac{125.134 - 124.942}{110 - 20}(C - 20)$$

$$\left[y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\right]$$

or $\quad L - 124.942 = \dfrac{0.192}{90}(C - 20)$

or $\quad L = \dfrac{0.192}{90}(C - 20) + 124.942.$

So, we have expressed L in terms of C.

17. The owner of a milk store finds that, he can sell 980 litres of milk each week at ₹ 14/litre and 1220 litres of milk each week at ₹ 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at ₹ 17/litre?

Sol. Assuming the price per litre, in rupees, along x-axis and the demand in litres, along y-axis, we have two points (14, 980) and (16, 1220) in xy-plane or PD-plane.

Using two-point form, the linear relationship between selling price and demand is

$$D - 980 = \frac{1220 - 980}{16 - 14}(P - 14) \qquad \Big|\ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $\quad D = 980 + \dfrac{240}{2}(P - 14)$ or $D = 980 + 120P - 1680$

or $\quad D = 120P - 700 \qquad \ldots(i)$

Equation (i) expresses D in terms of P.

When $P = 17$, equation (i) gives $\quad D = 120 \times 17 - 700$

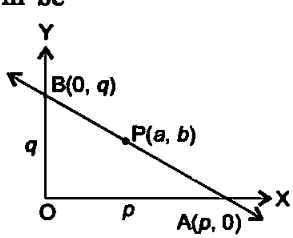
or $\quad D = 1340$

∴ The owner of the milk-store could sell 1340 litres of milk weekly at ₹ 17 per litre.

18. P(a, b) is the mid-point of a line segment between axes. Show that equation of the line is $\dfrac{x}{a} + \dfrac{y}{b} = 2.$

Sol. Let equation of line in intercept form be

$$\frac{x}{p} + \frac{y}{q} = 1 \qquad \ldots(i)$$



If it meets the x-axis in A and y-axis in B, then coordinates of A and B are $(p, 0)$ and $(0, q)$ respectively.

Since P(a, b) is mid-point of AB, we have

$$\left(\frac{p+0}{2}, \frac{0+q}{2}\right) = (a, b)$$

$\Rightarrow \quad \frac{p}{2} = a \quad$ and $\quad \frac{q}{2} = b$

$\Rightarrow \quad p = 2a \quad$ and $\quad q = 2b$

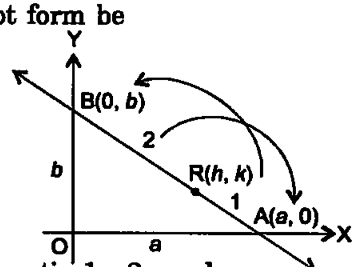
Putting these values of p and q in (i), the required equation of line is

$\frac{x}{2a} + \frac{y}{2b} = 1$ multiplying every term by 2, $\frac{x}{a} + \frac{y}{b} = 2.$

19. Point R(h, k) divides a line segment between the axes in the ratio 1 : 2. Find equation of the line.

Sol. Let equation of line in intercept form be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad ...(i)$$



If it meets the x-axis in A and y-axis in B, then coordinates of A and B are (a, 0) and (0, b) respectively.

Since R(h, k) divides AB in the ratio 1 : 2, we have

Using Section Formula,

$$\left(\frac{1\times 0+2a}{1+2}, \frac{1b+2\times 0}{1+2}\right) = (h, k)$$

$\Rightarrow \quad \frac{2a}{3} = h \quad$ and $\quad \frac{b}{3} = k \Rightarrow 2a = 3h$ and $b = 3k$

$\Rightarrow \quad a = \frac{3h}{2} \quad$ and $\quad b = 3k$

Putting these values of a and b in (i), the required equation of line is

$$\frac{x}{3h/2} + \frac{y}{3k} = 1 \quad \text{or} \quad \frac{2x}{3h} + \frac{y}{3k} = 1$$

Multiplying by $3hk$, we have

$$2kx + hy = 3hk.$$

20. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

Sol. Given points are A(3, 0), B(– 2, – 2) and C(8, 2).

Using two-point form, equation of line through A(3,0) and B(–2,–2) is

$$y - 0 = \frac{-2-0}{-2-3}(x-3)$$

or $$y = \frac{2}{5}(x-3) \quad \text{or} \quad 5y = 2x - 6$$

or $2x - 5y - 6 = 0$ ...(*i*)

The three points will be collinear if the third point C lies on it, *i.e.*, if the coordinates of C satisfy (*i*).

Putting $x = 8$ and $y = 2$ in (*i*), we have

$$2 \times 8 - 5 \times 2 - 6 = 0$$

or $16 - 10 - 6 = 0$ or $0 = 0$ which is true.

Hence, the three points are collinear.

EXERCISE 10.3 (*Page No.: 227–228*)

1. Reduce the following equations into slope-intercept form and find their slopes and the *y*-intercepts.

(*i*) $x + 7y = 0$, (*ii*) $6x + 3y - 5 = 0$, (*iii*) $y = 0$.

Sol. (*i*) Given equation $x + 7y = 0$ can be written as

$$7y = -x \quad \text{or} \quad y = -\frac{1}{7}x + 0$$

Comparing with $y = mx + c$, we have

slope $m = -\frac{1}{7}$, *y*-intercept $c = 0$.

(*ii*) Given equation $6x + 3y - 5 = 0$ can be written as $3y = -6x + 5$

Dividing by 3, $y = -2x + \frac{5}{3}$

Comparing with $y = mx + c$, we have

slope $m = -2$, *y*-intercept $c = \frac{5}{3}$.

(*iii*) Given equation $y = 0$ can be written as

$$y = 0x + 0$$

Comparing with $y = mx + c$, we have

slope $m = 0$, *y*-intercept $c = 0$.

2. Reduce the following equations into intercept form and find their intercepts on the axes.

(i) $3x + 2y - 12 = 0$, (ii) $4x - 3y = 6$,
(iii) $3y + 2 = 0$.

Sol. (i) Given equation $3x + 2y - 12 = 0$ can be written as

$$3x + 2y = 12$$

Dividing by 12 to make R.H.S. = 1, we have

or $\frac{3x}{12} + \frac{2y}{12} = 1$ or $\frac{x}{4} + \frac{y}{6} = 1$

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$, we have

x-intercept $a = 4$, y-intercept $b = 6$.

(ii) Given equation $4x - 3y = 6$ can be written as (on dividing by 6)

$$\frac{4x}{6} - \frac{3y}{6} = 1$$

or $\frac{2x}{3} - \frac{y}{2} = 1$

or $\frac{x}{3/2} + \frac{y}{-2} = 1$

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$, we have

x-intercept $a = \frac{3}{2}$, y-intercept $b = -2$.

(iii) Given equation $3y + 2 = 0$ can be written as

$$3y = -2 \quad \text{or} \quad y = -\frac{2}{3}$$

It is parallel to x-axis, therefore, there is no intercept with x-axis.

Intercept with y-axis $= -\frac{2}{3}$.

3. Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis.

(i) $x - \sqrt{3}\,y + 8 = 0$, (ii) $y - 2 = 0$, (iii) $x - y = 4$.

Sol. (i) Given equation is $x - \sqrt{3}\,y = -8$

or $-x + \sqrt{3}\,y = 8$ (Making R.H.S. + ve)

Dividing by $\sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$, we get

$$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 4$$

Comparing with $x \cos \omega + y \sin \omega = p$...(*i*), we get

$$\cos \omega = -\frac{1}{2}, \sin \omega = \frac{\sqrt{3}}{2}, p = 4$$

Here $\cos \omega$ is negative and $\sin \omega$ is positive.

$\Rightarrow$ ω lies in second quadrant.

$\cos \omega = -\cos 60° = \cos (180° - 60°) = \cos 120°$

$\Rightarrow$ $\omega = 120°$

Putting values of p and ω in (*i*),

$x \cos 120° + y \sin 120° = 4$ is the normal form and $p = 4$, $\omega = 120°$.

Remark: The interested reader, if he/she likes can find ω from $\sin \omega = \frac{\sqrt{3}}{2}$

(*ii*) Given equation is $y - 2 = 0$

or $0x + 1y = 2$ (R.H.S. + ve)

Dividing by $\sqrt{a^2+b^2} = \sqrt{0^2+1^2} = 1$, we get

$$0x + 1y = 2$$

Comparing with $x \cos \omega + y \sin \omega = p$...(*i*), we get

$\cos \omega = 0$, $\sin \omega = 1$, $p = 2$

$\Rightarrow$ $\omega = 90°$

Putting values of p and ω in (*i*),

$x \cos 90° + y \sin 90° = 2$ is the normal form and $p = 2$, $\omega = 90°$.

(*iii*) Given equation is $x - y = 4$ (R.H.S. + ve)

Dividing by $\sqrt{a^2+b^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 2\sqrt{2}$$

Comparing with $x \cos \omega + y \sin \omega = p$...(*i*), we get

$$\cos \omega = \frac{1}{\sqrt{2}}, \sin \omega = -\frac{1}{\sqrt{2}}, p = 2\sqrt{2}$$

$\Rightarrow$ ω lies in fourth quadrant. ($\because$ $\cos \omega$ is +ve and $\sin \omega$ is -ve)

$\cos \omega = \cos 45° = \cos (360° - 45°) = \cos 315°$

$\Rightarrow$ $\omega = 315°$

Putting values of p and ω in (*i*),

$x \cos 315° + y \sin 315° = 2\sqrt{2}$ is the normal form and $p = 2\sqrt{2}$, $\omega = 315°$.

4. Find the distance of the point (– 1, 1) from the line $12(x + 6) = 5(y - 2)$.

Sol. Equation of the given line is $12(x + 6) = 5(y - 2)$

or $\quad 12x + 72 = 5y - 10$

or $\quad 12x - 5y + 82 = 0$

(Perpendicular) distance of the point (– 1, 1) from this line

$$= \frac{|12(-1) - 5(1) + 82|}{\sqrt{(12)^2 + (-5)^2}} \qquad \left| \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \right.$$

$$= \frac{|-12 - 5 + 82|}{\sqrt{144 + 25}} = \frac{65}{13} = 5.$$

5. Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.

Sol. Let the coordinates of the required point on x-axis be P(x, 0).

Equation of the given line is $\frac{x}{3} + \frac{y}{4} = 1$

or $\quad 4x + 3y = 12 \quad$ or $\quad 4x + 3y - 12 = 0 \quad$ ...(*i*)

Perpendicular distance of the point P(x, 0) from line (*i*) is 4 (given)

$$\therefore \quad \frac{|4x + 3(0) - 12|}{\sqrt{(4)^2 + (3)^2}} = 4 \qquad \left| \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \right.$$

or $\quad \frac{|4x - 12|}{5} = 4 \quad$ or $\quad |4x - 12| = 20$

$\therefore \quad 4x - 12 = \pm 20 \quad | \because$ If $|x| = a$ and $a \geq 0$; then $x = \pm a$

Dividing by 4

$x - 3 = \pm 5 \quad$ or $\quad x = 3 \pm 5 = 8, -2$

$\therefore$ Required points are P(x, 0) = P(8, 0) and P(– 2, 0).

6. Find the distance between parallel lines

(*i*) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$

(*ii*) $l(x + y) + p = 0$ and $l(x + y) - r = 0$.

Sol. (*i*) Here A = 15, B = 8, $C_1 = -34$ and $C_2 = 31$.

$\therefore$ Required distance is

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}} = \frac{|-34 - 31|}{\sqrt{(15)^2 + (8)^2}} = \frac{|-65|}{\sqrt{289}} = \frac{65}{17}$$

(*ii*) Here A = l, B = l, $C_1 = p$ and $C_2 = -r$.

∴ Required distance is

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}} = \frac{|p - (-r)|}{\sqrt{l^2 + l^2}} = \frac{|p + r|}{\sqrt{2l^2}}$$

$$= \frac{|p + r|}{\sqrt{2}\,|l|} \qquad [\because \sqrt{l^2} = |l|]$$

$$= \frac{1}{\sqrt{2}}\left|\frac{p + r}{l}\right|. \qquad \left[\because \frac{|x|}{|y|} = \left|\frac{x}{y}\right|\right]$$

7. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (– 2, 3).

Sol. Given line is $3x - 4y + 2 = 0$.

Its slope $= -\frac{3}{-4} = \frac{3}{4}$ $\qquad \left[m = -\frac{a}{b}\right]$

Required line is parallel to the given line.

∴ Slope of required line $= \frac{3}{4}$ $\qquad [m_1 = m_2]$

It passes through the point (– 2, 3).

∴ Equation of required line is

$$y - 3 = \frac{3}{4}[x - (-2)] \qquad \text{[Point-slope form]}$$

or $4y - 12 = 3x + 6$

or $3x - 4y + 18 = 0$.

8. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x-intercept 3.

Sol. Given line is $x - 7y + 5 = 0$.

Its slope $= -\frac{1}{-7} = \frac{1}{7}$ $\qquad \left[m = -\frac{a}{b}\right]$

Required line is perpendicular to given line.

∴ Slope of required line = – 7. $\qquad$ [– ve reciprocal]

It has x-intercept 3, therefore, it passes through the point (3, 0).

∴ Equation of line with slope – 7 and passing through the point (3, 0) is

$$y - 0 = -7(x - 3) \qquad \text{[Point-slope form]}$$

or $y = -7x + 21$ or $7x + y = 21$.

9. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.

Sol. Slopes of lines are

$$m_1 = \frac{-a}{b} = -\frac{\sqrt{3}}{1} = -\sqrt{3} \quad \text{and} \quad m_2 = -\frac{1}{\sqrt{3}}$$

The acute angle θ between the two lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\sqrt{3} + \frac{1}{\sqrt{3}}}{1 + (-\sqrt{3})\left(-\frac{1}{\sqrt{3}}\right)}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\frac{-3+1}{\sqrt{3}}}{1+1}\right| = \frac{\frac{2}{\sqrt{3}}}{2} = \frac{1}{\sqrt{3}} = \tan 30°$$

which gives θ = 30°. Hence, angle between the two lines is either 30° or 180° – 30° = 150°.

Remark: We know that there are two angles between two lines θ and 180° – θ (linear pair).

10. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$ at right angles. Find the value of h.

Sol. Slope of line L_1 through the points $(h, 3)$ and $(4, 1)$ is

$$m_1 = \frac{1-3}{4-h} = \frac{-2}{4-h}$$

Slope of line L_2: $7x - 9y - 19 = 0$ is

$$m_2 = -\frac{a}{b} = -\frac{7}{-9} = \frac{7}{9}$$

Since $L_1 \perp L_2$, therefore, $m_1 m_2 = -1$

$$\Rightarrow \quad \left(\frac{-2}{4-h}\right)\left(\frac{7}{9}\right) = -1 \text{ or } \frac{-14}{36-9h} = -1 \quad \text{or } -14 = -36 + 9h$$

or $\quad 9h = 22 \qquad \therefore \quad h = \frac{22}{9}$.

11. Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is

$$\mathbf{A(x - x_1) + B(y - y_1) = 0.}$$

Sol. Given line is $Ax + By + C = 0$.

Its slope $= -\dfrac{A}{B}$.

Required line is parallel to the given line.

$\therefore$ Slope of required line $= -\dfrac{A}{B}$.

$\therefore$ Equation of the line through the point (x_1, y_1) and having slope $-\dfrac{A}{B}$ is

$$y - y_1 = -\frac{A}{B}(x - x_1)$$

or $$B(y - y_1) = -A(x - x_1)$$

or $A(x - x_1) + B(y - y_1) = 0$.

12. Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If slope of one line is 2; find equation of the other line.

Sol. Slope of one line is $m_1 = 2$. Let the slope of the other line be m_2.

Since angle between the lines is 60°, we have

$$\tan 60^\circ = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

or $$\sqrt{3} = \left|\frac{2 - m_2}{1 + 2m_2}\right| \quad \text{or} \quad \frac{2 - m_2}{1 + 2m_2} = \pm\sqrt{3}$$

Taking + ve sign, we have

$$2 - m_2 = \sqrt{3}(1 + 2m_2)$$

or $$2 - m_2 = \sqrt{3} + 2\sqrt{3}\, m_2$$

or $$(1 + 2\sqrt{3})m_2 = 2 - \sqrt{3}$$

$\therefore$ $$m_2 = \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}}$$

$\therefore$ Equation of line through the point (2, 3) and having slope $\dfrac{2 - \sqrt{3}}{1 + 2\sqrt{3}}$ is

$$y - 3 = \frac{2 - \sqrt{3}}{1 + 2\sqrt{3}}(x - 2)$$

or $(1 + 2\sqrt{3})y - 3 - 6\sqrt{3} = (2 - \sqrt{3})x - 4 + 2\sqrt{3}$

or $(\sqrt{3} - 2)x + (1 + 2\sqrt{3})y = -1 + 8\sqrt{3}$.

Taking – ve sign, we have

$$2 - m_2 = -\sqrt{3}(1 + 2m_2)$$

or $$2 - m_2 = -\sqrt{3} - 2\sqrt{3}\,m_2$$

or $$\sqrt{3} + 2 = (1 - 2\sqrt{3})m_2$$

$\therefore$ $$m_2 = \frac{\sqrt{3}+2}{1-2\sqrt{3}}$$

$\therefore$ Equation of line through the point (2, 3) and having slope $\frac{\sqrt{3}+2}{1-2\sqrt{3}}$ is

$$y - 3 = \frac{\sqrt{3}+2}{1-2\sqrt{3}}(x - 2)$$

or $$(1 - 2\sqrt{3})y - 3 + 6\sqrt{3} = (\sqrt{3} + 2)x - 2\sqrt{3} - 4$$

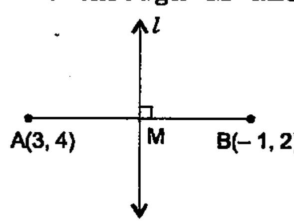
or $$(\sqrt{3} + 2)x + (2\sqrt{3} - 1)y = 8\sqrt{3} + 1.$$

13. Find the equation of the right bisector of the line segment joining the points (3, 4) and (– 1, 2).

Sol. Let A(3, 4) and B(– 1, 2) be the two given points.

Mid-point of AB is $M = \left(\frac{3-1}{2}, \frac{4+2}{2}\right) = (1, 3)$.

Right bisector of AB is the line l through M and perpendicular to AB.



Slope of AB $= \frac{2-4}{-1-3} = \frac{-2}{-4} = \frac{1}{2}$

Since $l \perp$ AB, slope of $l = -2$.

$\therefore$ Equation of right bisector l is

$$y - 3 = -2(x - 1)$$

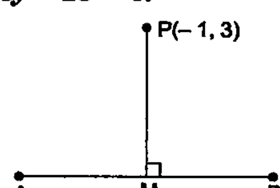
or $y - 3 = -2x + 2$ or $2x + y = 5$.

14. Find the coordinates of the foot of perpendicular from the point (– 1, 3) to the line 3x – 4y – 16 = 0.

Sol. Given line is AB: $3x - 4y - 16 = 0$.

Given point is P(– 1, 3).

Draw PM $\perp$ AB. We have to find the coordinates of M.



Slope of AB $= -\frac{3}{-4} = \frac{3}{4}$.

Since PM $\perp$ AB, slope of PM $= -\frac{4}{3}$ (– ve reciprocal)

Using point-slope form, equation of PM is

$$y - 3 = -\frac{4}{3}(x - (-1))$$

or $3y - 9 = -4x - 4$ or $4x + 3y - 5 = 0$

Now AB: $3x - 4y - 16 = 0$ ... (*i*)

PM: $4x + 3y - 5 = 0$ ... (*ii*)

Now foot M of perpendicular is the point of intersection of lines (*i*) and (*ii*). **So let us solve them for *x* and *y*.**

Eqn(*i*) × 3 + Eqn(*ii*) × 4 gives to eliminate *y*,

$9x - 12y - 48 + 16x + 12y - 20 = 0$

$$\Rightarrow 25x - 68 = 0 \Rightarrow 25x = 68 \Rightarrow x = \frac{68}{25}.$$

Putting $x = \frac{68}{25}$ *in* (*i*), $\frac{204}{25} - 4y - 16 = 0$

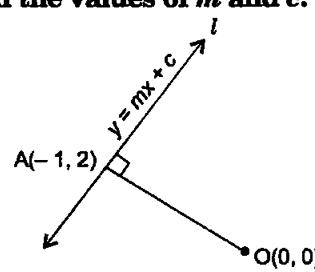
$$\Rightarrow -4y = 16 - \frac{204}{25} = \frac{400-204}{25} = \frac{196}{25}$$

$$\therefore y = \frac{-196}{4\times 25} = \frac{-49}{25}$$

$\therefore$ co-ordinates of foot of perpendicular M are $M\left(\frac{68}{25}, \frac{-49}{25}\right)$.

15. The perpendicular from the origin to the line $y = mx + c$ meets it at the point (– 1, 2). Find the values of *m* and *c*.

Sol. Let OA be perpendicular from the origin O(0, 0) to the line *l* whose equation is $y = mx + c$. (slope – intercept form)



Slope of line $l = m$.

Slope of OA $= \frac{2-0}{-1-0} = -2$.

Since $l \perp OA$, we have $(m)(-2) = -1$ $\quad | m_1m_2 = -1 |$

$$\Rightarrow -2m = -1 \text{ or } m = \frac{1}{2}$$

$\therefore$ Equation of line *l* becomes

$$y = \frac{1}{2}x + c$$

The point A(– 1, 2) lies on it

$$\therefore \quad 2 = \frac{1}{2}(-1) + c \text{ or } 2 = \frac{-1}{2} + c \text{ or } c = 2 + \frac{1}{2} = \frac{5}{2}$$

Hence $m = \frac{1}{2}, c = \frac{5}{2}$.

16. If p and q are the lengths of perpendiculars from the origin to the lines $x \cos \theta - y \sin \theta = k \cos 2\theta$ and $x \sec \theta + y \operatorname{cosec} \theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$.

Sol. Equations of the given lines are

$x \cos \theta - y \sin \theta = k \cos 2\theta$ or $x \cos \theta - y \sin \theta - k \cos 2\theta = 0$ ...(*i*)

and $x \sec \theta + y \operatorname{cosec} \theta = k$ or $x \sec \theta + y \operatorname{cosec} \theta - k = 0$ ...(*ii*)

p = length of ⊥ from origin (0, 0) on line (*i*)

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|0 - 0 - k \cos 2\theta|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = | k \cos 2\theta | \quad ...(iii)$$

q = Length of ⊥ from origin (0, 0) on line (*ii*)

$$= \frac{|0 + 0 - k|}{\sqrt{\sec^2 \theta + \operatorname{cosec}^2 \theta}} = \frac{|k|}{\sqrt{\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}}}$$

$$= \frac{|k|}{\sqrt{\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}}}$$

or $q = | k | \cos \theta \sin \theta$ ...(*iv*)

Putting values of p and q from (*iii*) and (*iv*) in L.H.S. $= p^2 + 4q^2$, we have

L.H.S. $= k^2 \cos^2 2\theta + 4 k^2 \cos^2 \theta \sin^2 \theta$

$= k^2 \cos^2 2\theta + k^2 (2 \sin \theta \cos \theta)^2$

$= k^2 \cos^2 2\theta + k^2 \sin^2 2\theta$

$= k^2 (\cos^2 2\theta + \sin^2 2\theta) = k^2 =$ R.H.S.

17. In the triangle ABC with vertices A(2, 3), B(4, – 1) and C(1, 2), find the equation and length of altitude from the vertex A.

Sol. Slope of BC $= \frac{2 - (-1)}{1 - 4} = \frac{3}{-3} = -1$.

Draw AD ⊥ BC.

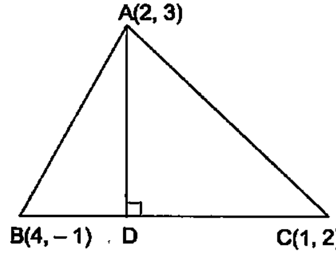
∴ Slope of altitude AD = 1. (–ve reciprocal)

Using point-slope form, equation of altitude AD is

$$y - 3 = 1(x - 2)$$

or $\quad y - 3 = x - 2$

or $\quad y - x = 1$



Equation of BC is

$$y - (-1) = \frac{2-(-1)}{1-4}(x - 4)$$

[Two-point form]

or $y + 1 = \frac{3}{-3}(x - 4)$ or $y + 1 = -(x - 4)$

or $y + 1 = -x + 4$

or $\quad x + y - 3 = 0$

Length of altitude AD

= Length of perpendicular form A(2, 3) on BC

$$= \frac{|2+3-3|}{\sqrt{1^2+1^2}} \qquad \left| \; d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \right.$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}.$$

18. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Sol. We know that equation of the line whose intercepts on the axes are a and b is $\frac{x}{a} + \frac{y}{b} = 1$ or $\frac{x}{a} + \frac{y}{b} - 1 = 0$

Length of the perpendicular segment from the origin (0, 0) on this line is p. (given)

$$\therefore \quad p = \frac{|0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \qquad \left| \; \frac{|ax_1 + by_1 + c_1|}{\sqrt{a^2+b^2}} \right.$$

Squaring both sides, $\quad p^2 = \dfrac{1}{\frac{1}{a^2}+\frac{1}{b^2}}$

Cross-multiplying $\quad p^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = 1$

Dividing by p^2, $\quad \frac{1}{a^2}+\frac{1}{b^2} = \frac{1}{p^2}.$

Or $\quad \frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}.$

MISCELLANEOUS EXERCISE ON CHAPTER 10

(*Page No.: 233–234*)

1. Find the values of k for which the line

$$(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0 \text{ is}$$

(*i*) parallel to the x-axis

(*ii*) parallel to the y-axis

(*iii*) passing through the origin.

Sol. Given line is

$$(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0 \quad \text{...}(i)$$

(*i*) If the line (*i*) is parallel to the x-axis, then its equation must be of the form $y = b$.

i.e., the term of x must be absent

i.e., the coefficient of x must be zero.

$\therefore \quad k - 3 = 0$ or $k = 3$.

(*ii*) If the line (*i*) is parallel to the y-axis, then its equation must be of the form $x = a$.

i.e., the term of y must be absent

i.e., the coefficient of y must be zero.

$\therefore \quad -(4 - k^2) = 0$ or $-4 + k^2 = 0$ or $k^2 = 4 \quad \therefore k = \pm 2$.

(*iii*) If the line (*i*) passes through the origin (0, 0), then

$x = 0, y = 0$ satisfy (*i*).

i.e., $0 - 0 + k^2 - 7k + 6 = 0$

i.e., $(k - 1)(k - 6) = 0 \quad \therefore \quad k = 6$ or 1.

2. Find the values of θ and p, if the equation $x \cos \theta + y \sin \theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$.

Sol. Given equation is $\sqrt{3}x + y + 2 = 0 \Rightarrow \sqrt{3}x + y = -2$

Multiplying by -1, $-\sqrt{3}x - y = 2$ (Making R.H.S. + ve)

Dividing by $\sqrt{a^2 + b^2} = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$, we get

$$-\frac{\sqrt{3}}{2}x - \frac{1}{2}y = 1$$

Comparing with $x \cos \theta + y \sin \theta = p$, we get

$$p = 1, \cos\theta = -\frac{\sqrt{3}}{2}, \sin\theta = -\frac{1}{2}$$

Since cos θ and sin θ both are – ve, θ lies in the 3rd quadrant.

$$\therefore \quad \cos\theta = -\frac{\sqrt{3}}{2} = -\cos 30°$$
$$= \cos(180° + 30°) = \cos 210°$$
$$\Rightarrow \quad \theta = 210° = \frac{7\pi}{6}$$

Hence, $\theta = \frac{7\pi}{6}$ and $p = 1$.

Note: The interested reader may find θ from $\sin\theta = \frac{-1}{2}$.

3. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and – 6, respectively.

Sol. Let the intercepts of the required line on the axes be *a* and *b*.

∴ Equation of the required line is $\frac{x}{a} + \frac{y}{b} = 1$ ...(*i*)

According to given sum of intercepts $= a + b = 1$ ...(*ii*)

and product of intercepts $= ab = -6$ ...(*iii*)

Let us solve equations (*ii*) and (*iii*) for *a* and *b*.

From equation (*ii*), $b = 1 - a$ ...(*iv*)

Putting this value of *b* in equation (*iii*),

$$a(1 - a) = -6 \quad \text{or} \quad a - a^2 + 6 = 0$$

or $-a^2 + a + 6 = 0$ or $a^2 - a - 6 = 0$ or $(a - 3)(a + 2) = 0$

∴ Either $a = 3$ or $a = -2$

If $a = 3$, from equation (*iv*), $b = 1 - a = 1 - 3 = -2$

Putting these values of *a* and *b* in (*i*), equation of one required line is

$$\frac{x}{3} + \frac{y}{-2} = 1 \quad \text{or} \quad \frac{x}{3} - \frac{y}{2} = 1 \text{ or } \frac{2x - 3y}{6} = 1$$

or $2x - 3y = 6$

If $a = -2$, from equation (*iv*), $b = 1 - a = 1 + 2 = 3$

Putting these values of *a* and *b* in (*i*), equation of second required line is $\frac{x}{-2} + \frac{y}{3} = 1$.

Multiplying by – 6, $3x - 2y = -6$ or $3x - 2y + 6 = 0$.

4. What are the points on the y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units?

Sol. Any point on y-axis is $(0, y)$. If its distance from the line

$$\frac{x}{3} + \frac{y}{4} = 1 \text{ i.e., } \frac{4x+3y}{12} = 1 \Rightarrow 4x + 3y = 12$$

or $4x + 3y - 12 = 0$ is 4 units, then

$$\frac{|4 \times 0 + 3 \times y - 12|}{\sqrt{4^2 + 3^2}} = 4$$

$$\Rightarrow \frac{|3y - 12|}{5} = 4 \qquad \Rightarrow |3y - 12| = 20$$

$$\Rightarrow 3y - 12 = \pm 20 \qquad |\because |x| = a\ (a \geq 0) \Rightarrow x \pm a$$

$$\Rightarrow 3y = \pm 20 + 12 \qquad \Rightarrow 3y = 32 \text{ or } -8$$

$$\Rightarrow y = \frac{32}{3} \text{ or } -\frac{8}{3}$$

Hence, the required points on y-axis are $(0, y) = \left(0, \frac{32}{3}\right)$ and $\left(0, -\frac{8}{3}\right)$.

5. Find perpendicular distance from the origin of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Sol. Equation of line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ is (Two point form)

$$y - \sin\theta = \frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta}(x - \cos\theta)$$

or
$$y - \sin\theta = \frac{2\cos\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}}{-2\sin\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}}(x - \cos\theta)$$

or
$$y - \sin\theta = -\frac{\cos\frac{\phi+\theta}{2}}{\sin\frac{\phi+\theta}{2}}(x - \cos\theta)$$

cross-multiplying

$$(y - \sin\theta)\sin\frac{\phi+\theta}{2} = -\cos\frac{\phi+\theta}{2}(x - \cos\theta)$$

or
$$y\sin\frac{\phi+\theta}{2} - \sin\theta\sin\frac{\phi+\theta}{2} = -x\cos\frac{\phi+\theta}{2} + \cos\theta\cos\frac{\phi+\theta}{2}$$

or $\quad x \cos \frac{\phi+\theta}{2} + y \sin \frac{\phi+\theta}{2} - \left(\cos\theta \cos\frac{\phi+\theta}{2} + \sin\theta \sin\frac{\phi+\theta}{2}\right) = 0$

or $\quad x \cos \frac{\phi+\theta}{2} + y \sin \frac{\phi+\theta}{2} - \cos\left(\theta - \frac{\phi+\theta}{2}\right) = 0$

[∵ cos A cos B + sin A sin B = cos(A–B)]

or $\quad x \cos \frac{\phi+\theta}{2} + y \sin \frac{\phi+\theta}{2} - \cos \frac{\theta-\phi}{2} = 0$

$$\left[\because \theta - \left(\frac{\theta+\phi}{2}\right) = \frac{2\theta-\theta-\phi}{2} = \frac{\theta-\phi}{2}\right]$$

Its distance from origin (0, 0) is $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$$= \frac{\left|0 + 0 - \cos\frac{\theta-\phi}{2}\right|}{\sqrt{\cos^2\frac{\phi+\theta}{2} + \sin^2\frac{\phi+\theta}{2}}} = \left|\cos\frac{\theta-\phi}{2}\right| \quad (\because |-t| = |t|)$$

Note 1. The answer may be written as

$$\left|\cos\frac{\phi-\theta}{2}\right| \qquad [\because \cos(-x) = \cos x]$$

$$= \left|\frac{2\sin\frac{\phi-\theta}{2}\cos\frac{\phi-\theta}{2}}{2\sin\frac{\phi-\theta}{2}}\right|$$

$$= \frac{|\sin(\phi-\theta)|}{\left|2\sin\frac{\phi-\theta}{2}\right|} \qquad [\because 2\sin x\cos x = \sin 2x]$$

$$= \frac{|\sin(\phi-\theta)|}{2\left|\sin\frac{\phi-\theta}{2}\right|}.$$

Second solution:

Equation of line is

$$y - \sin\theta = \frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta}(x - \cos\theta)$$

or $\quad y(\cos\phi - \cos\theta) - \sin\theta\cos\phi + \sin\theta\cos\theta$

$$= x(\sin\phi - \sin\theta) - \sin\phi\cos\theta + \sin\theta\cos\theta$$

or $(\sin\phi - \sin\theta)x - (\cos\phi - \cos\theta)y - \sin\phi\cos\theta + \cos\phi\sin\theta = 0$

or $\sin\phi - \sin\theta)x - (\cos\phi - \cos\theta)y - \sin(\phi - \theta) = 0$

Its distance from origin (0, 0) is

$$\frac{|0 - 0 - \sin(\phi - \theta)|}{\sqrt{(\sin\phi - \sin\theta)^2 + (\cos\phi - \cos\theta)^2}}$$

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{(\sin^2\phi + \cos^2\phi) + (\sin^2\theta + \cos^2\theta) - 2(\cos\phi\cos\theta + \sin\phi\sin\theta)}}$$

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{1 + 1 - 2\cos(\phi - \theta)}} = \frac{|\sin(\phi - \theta)|}{\sqrt{2[1 - \cos(\phi - \theta)]}}$$

$$= \frac{|\sin(\phi - \theta)|}{\sqrt{2 \times 2\sin^2\frac{\phi - \theta}{2}}} = \frac{|\sin(\phi - \theta)|}{2\sqrt{\sin^2\frac{\phi - \theta}{2}}}$$

$$= \frac{|\sin(\phi - \theta)|}{2\left|\sin\frac{\phi - \theta}{2}\right|}. \qquad [\because \sqrt{x^2} = |x|]$$

6. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.

Sol. To find the point of intersection of the lines

$$x - 7y + 5 = 0 \qquad ...(i)$$

and $$3x + y = 0 \qquad ...(ii)$$

let us solve (*i*) and (*ii*) for x and y.

eqn(*i*) + 7 × eqn(*ii*) gives (To eliminate y),

$x - 7y + 5 + 21x + 7y = 0 \Rightarrow 22x + 5 = 0$

$\Rightarrow 22x = -5 \Rightarrow x = -\frac{5}{22}$

Putting $x = -\frac{5}{22}$ in (*ii*), $-\frac{15}{22} + y = 0$ or $y = \frac{15}{22}$

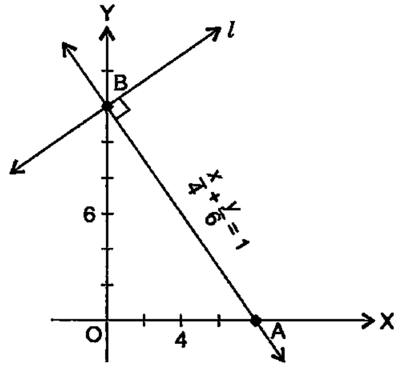
∴ The point of intersection of given lines is

$$\left(-\frac{5}{22}, \frac{15}{22}\right) = (x_1, y_1).$$

Equation of line through this point and parallel to y-axis is

$x = x_1$, *i.e.*, $\qquad x = -\frac{5}{22}$.

7. Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point where it meets the y-axis.



Sol. The given line is $\frac{x}{4}+\frac{y}{6}=1$

or $\frac{3x+2y}{12}=12$

or $3x + 2y = 12$ ...(*i*)

Its slope $= -\frac{3}{2}$

Required line l is perpendicular to the given line.

$\therefore$ Slope of $l = \frac{2}{3}$. (– ve reciprocal)

Also the line (*i*) meets y-axis ($x = 0$) at B(0, 6).

(Because putting $x = 0$ in (*i*), $2y = 12 \Rightarrow y = 6$)

Required line l passes through B(0, 6) and has slope $\frac{2}{3}$.

$\therefore$ Equation of l is

$$y - 6 = \frac{2}{3}(x - 0) \qquad \text{[Point-slope form]}$$

or $3y - 18 = 2x$ or $-2x + 3y - 18 = 0$

or $2x - 3y + 18 = 0$.

8. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.

Sol. The equations of the sides of triangle ABC (say) are

BC: $y - x = 0$ ...(*i*)

CA: $x + y = 0$ ...(*ii*)

AB: $x - k = 0$ ...(*iii*)

To find vertex A, let us solve (*ii*) and (*iii*) for x and y.

From (*iii*), $x = k$.

Putting $x = k$ *in* (*ii*), $k + y = 0$ $\therefore y = -k$

$\therefore$ vertex A is $(k, -k) = (x_1, y_1)$ (say)

To find vertex B, let us solve (*i*) and (*iii*) for x and y.

From (*iii*), $x = k$

Putting $x = k$ *in* (*i*), $y - k = 0$. $\therefore y = k$

$\therefore$ vertex B is $(k, k) = (x_2, y_2)$ (say)

To find vertex C, let us solve (*i*) and (*ii*) for x and y.

Adding (*i*) and (*ii*), $2y = 0$ or $y = \frac{0}{2} = 0$.

Putting $y = 0$ in (*ii*), $x = 0$

$\therefore$ vertex C is $(0, 0) = (x_3, y_3)$ (say)

Here $x_1 = k, y_1 = -k; x_2 = k, y_2 = k; x_3 = 0, y_3 = 0$.

$\therefore$ Area of triangle ABC

$$= \frac{1}{2} | x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) |$$

$$= \frac{1}{2} | k(k - 0) + k(0 + k) + 0(-k - k) |$$

$$= \frac{1}{2} | k^2 + k^2 | = \frac{1}{2} | 2k^2 | = \frac{1}{2} (2k^2) = k^2 \text{ sq. units.}$$

9. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

Sol. Equations of the three lines are

$$3x + y - 2 = 0 \quad ...(i)$$
$$px + 2y - 3 = 0 \quad ...(ii)$$
$$2x - y - 3 = 0 \quad ...(iii)$$

To find the point of intersection of any two lines, say (*i*) and (*iii*). [because p is absent in both(*i*) and (*iii*)]

Adding $5x - 5 = 0$ or $5x = 5$ or $x = 1$.

Putting $x = 1$ in (*i*), $3 + y - 2 = 0$ or $y + 1 = 0$ or $y = -1$

$\therefore$ Point of intersection of (*i*) and (*iii*) is $(1, -1)$.

It lies on (*ii*). ($\because$ three lines intersect at one point)

$\therefore$ $p(1) + 2(-1) - 3 = 0 \Rightarrow p - 2 - 3 = 0 \Rightarrow p = 5$.

10. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that

$$m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$$

Sol. The equations of the three lines are

$$y = m_1x + c_1 \quad ...(i)$$
$$y = m_2x + c_2 \quad ...(ii)$$
$$y = m_3x + c_3 \quad ...(iii)$$

To find the point of intersection of any two lines, say (*ii*) and (*iii*).

Subtracting (*iii*) from (*ii*), $0 = (m_2 - m_3)x + c_2 - c_3$

or $c_3 - c_2 = (m_2 - m_3)x$

$$\therefore \quad x = \frac{c_3 - c_2}{m_2 - m_3}$$

Putting in (*ii*),

$$y = m_2\left(\frac{c_3 - c_2}{m_2 - m_3}\right) + c_2 = \frac{m_2(c_3 - c_2) + (m_2 - m_3)\,c_2}{m_2 - m_3}$$

$$= \frac{m_2c_3 - m_2c_2 + m_2c_2 - m_3c_2}{m_2 - m_3} = \frac{m_2c_3 - m_3c_2}{m_2 - m_3}$$

∴ Point of intersection of line (*ii*) and (*iii*) is

$$P\left(\frac{c_3 - c_2}{m_2 - m_3}, \frac{m_2c_3 - m_3c_2}{m_2 - m_3}\right)$$

Since the three lines are concurrent, P lies on (*i*).

$$\Rightarrow \quad \frac{m_2c_3 - m_3c_2}{m_2 - m_3} = m_1\left(\frac{c_3 - c_2}{m_2 - m_3}\right) + c_1$$

$$= \frac{m_1(c_3 - c_2) + c_1(m_2 - m_3)}{m_2 - m_3}$$

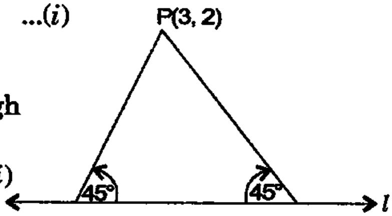
$$\Rightarrow \quad m_2c_3 - m_3c_2 = m_1c_3 - m_1c_2 + m_2c_1 - m_3c_1$$

$$\Rightarrow \quad m_1c_2 - m_1c_3 + m_2c_3 - m_2c_1 + m_3c_1 - m_3c_2 = 0$$

$$\Rightarrow \quad m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0.$$

11. Find the equations of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.

Sol. Given line is l: $x - 2y = 3$ ...(*i*)



Its slope $= \frac{1}{2}$

Equation of any line through P(3, 2) is

$$y - 2 = m(x - 3) \quad ...(ii)$$

where m is the slope of line.

Since the angle between (*i*) and (*ii*) is 45°.

We have

$$\tan 45° = \left|\frac{m - \frac{1}{2}}{1 + m.\frac{1}{2}}\right| = \left|\frac{\frac{2m-1}{2}}{\frac{2+m}{2}}\right| \quad \left[\because \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|\right]$$

$$\Rightarrow \quad 1 = \left|\frac{2m-1}{m+2}\right| \quad \Rightarrow \quad \frac{2m-1}{m+2} = \pm 1.$$

Taking + ve sign, $2m - 1 = m + 2$

$$\Rightarrow \quad m = 3$$

Putting this value of m in (*ii*), the equation of one line is

$$y - 2 = 3(x - 3) \quad \text{or} \quad 3x - y = 7$$

Taking – ve sign, $2m - 1 = -m - 2$

or $3m = -1$ or $m = -\frac{1}{3}$

Putting this value of m in (*ii*), the equation of second line is

$$y - 2 = -\frac{1}{3}(x - 3) \quad \text{or} \quad 3y - 6 = -x + 3$$

or $x + 3y = 9$

Hence, the equations of two required lines are

$$3x - y = 7,\ x + 3y = 9.$$

Remark: Whenever you are asked to find equations of lines (i.e., plural form), start thinking and doing to find the equation of a line (singular form), you will get the required lines.

12. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

Sol. Given lines are $4x + 7y - 3 = 0$ ...(*i*)

and $2x - 3y + 1 = 0$ ...(*ii*)

To find point of intersection of lines (*i*) and (*ii*), solve them for x and y.

eqn(*i*) × 3 + eqn(*ii*) × 7 gives to eliminate y,

$12x + 21y - 9 + 14x - 21y + 7 = 0$

$\Rightarrow 26x - 2 = 0 \Rightarrow 26x = 2 \Rightarrow x = \frac{2}{26} = \frac{1}{13}$

Putting $x = \frac{1}{13}$ in (*i*), $\frac{4}{13} + 7y - 3 = 0$

$\Rightarrow 7y = 3 - \frac{4}{13} = \frac{39-4}{13} = \frac{35}{13}$

$\Rightarrow y = \frac{35}{7 \times 13} = \frac{5}{13}$

∴ The point of intersection of given lines is $P\left(\frac{1}{13}, \frac{5}{13}\right)$.

Required line has equal intercepts on the axes.

Let each intercept be a, the equation of line is

$$\frac{x}{a} + \frac{y}{a} = 1 \quad \Rightarrow \frac{x+y}{a} = 1$$

or $x + y = a$ ...(*iii*)

It passes through $P\left(\frac{1}{13}, \frac{5}{13}\right)$.

[Therefore, putting $x = \frac{1}{13}, y = \frac{5}{13}$ in (*iii*),

We have $\frac{1}{13} + \frac{5}{13} = a$ or $a = \frac{6}{13}$

Putting $a = \frac{6}{13}$ in (*iii*), the required line is

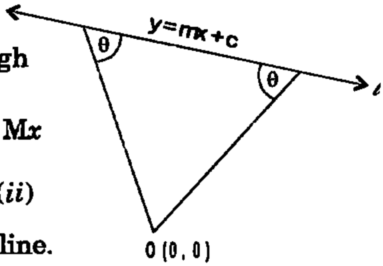
$$x + y = \frac{6}{13} \quad \text{or} \quad 13x + 13y = 6.$$

13. Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$.

Sol. Given line is

$l: y = mx + c$ (slope – intercept form) ...(*i*)

Its slope = m.



Equation of any line through the origin O(0, 0) is

$$y - 0 = M(x - 0) \Rightarrow y = Mx$$

or $\frac{y}{x} = M$ ...(*ii*)

where M is the slope of the line.

Since the angle between (*i*) and (*ii*) is θ, we have

$$\tan\theta = \left|\frac{M - m}{1 + Mm}\right|$$

or $$\tan\theta = \pm\frac{M - m}{1 + Mm}$$

Taking + ve sign

$$\tan\theta = \frac{M - m}{1 + Mm}$$

cross-multiplying $\tan\theta\,(1 + Mm) = M - m$

$\Rightarrow \tan\theta + Mm\tan\theta = M - m \Rightarrow m + \tan\theta = M - Mm\tan\theta$

$\Rightarrow m + \tan\theta = M(1 - m\tan\theta)$

$\Rightarrow$ $$M = \frac{m + \tan\theta}{1 - m\tan\theta}$$

Taking – ve sign

$$\tan\theta = -\frac{M - m}{1 + Mm}$$

$\Rightarrow \tan\theta\,(1 + Mm) = -M + m$

$\Rightarrow \tan\theta + Mm\tan\theta = -M + m \Rightarrow M + Mm\tan\theta = m - \tan\theta$

$\Rightarrow M(1 + m\tan\theta) = m - \tan\theta$

$\Rightarrow$ $$M = \frac{m - \tan\theta}{1 + m\tan\theta}$$

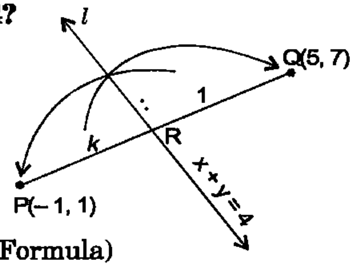
Combining the two values of M, we have

$$M = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$$

Putting this value of M in (*ii*),
the required equation of line is

$$\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}.$$

14. In what ratio, the line joining (– 1, 1) and (5, 7) is divided by the line $x + y = 4$?



Sol. Let the line
$l: x + y = 4$ ...(*i*)
meet the line joining P(– 1, 1) and Q(5, 7) at R. Let R divide PQ in the ratio $k : 1$, then

$$R = \left(\frac{5k-1}{k+1}, \frac{7k+1}{k+1}\right) \text{ (By Section Formula)}$$

Since R lies on line l, therefore, the coordinates of R satisfy equation (*i*)

$$\Rightarrow \frac{5k-1}{k+1} + \frac{7k+1}{k+1} = 4 \Rightarrow \frac{5k-1+7k+1}{k+1} = 4 \Rightarrow \frac{12k}{k+1} = 4$$

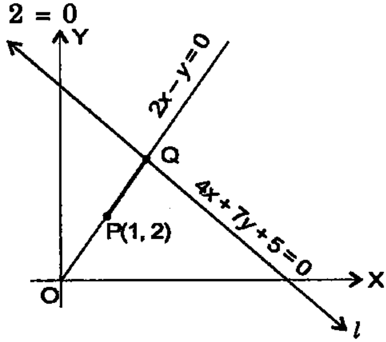
$$\Rightarrow 12k = 4k + 4$$

$$\Rightarrow 8k = 4 \quad \Rightarrow \quad k = \frac{1}{2}$$

Hence, the line joining P and Q is divided by the line l in the ratio $\frac{1}{2} : 1$, *i.e.*, 1 : 2.

15. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.

Sol. The point P(1, 2) lies on the line $2x - y = 0$, since $2 \times 1 - 2 = 0$. *i.e.*, $2 - 2 = 0$



To find the required distance PQ, we first find Q, the point of intersection of the lines

$4x + 7y + 5 = 0$ ...(*i*)
and $2x - y = 0$ ...(*ii*)

To find point Q, point of intersection of lines (*i*) and (*ii*), let us solve (*i*) and (*ii*) for x and y.

From (*ii*) $2x = y$, i.e., $y = 2x$

Putting $y = 2x$ in (*i*), $4x + 14x + 5 = 0 \Rightarrow 18x = -5$

$$\Rightarrow x = -\frac{5}{18} \text{ and therefore } y = 2x = -\frac{10}{18} = -\frac{5}{9}$$

$$\therefore \text{ Point Q} = \left(-\frac{5}{18}, -\frac{5}{9}\right).$$

$$\therefore \text{ Required distance} = PQ = \sqrt{\left(-\frac{5}{18}-1\right)^2 + \left(-\frac{5}{9}-2\right)^2}$$

$$= \sqrt{\left(\frac{-5-18}{18}\right)^2 + \left(\frac{-5-18}{9}\right)^2} = \sqrt{\left(\frac{-23}{18}\right)^2 + \left(\frac{-23}{9}\right)^2}$$

$$= \sqrt{\frac{529}{324} + \frac{529}{81}} = \sqrt{\frac{529}{81}\left(\frac{1}{4}+1\right)}$$

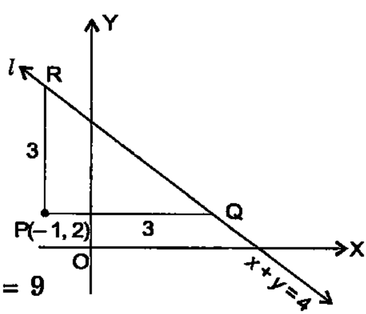
$$= \sqrt{\frac{529}{81} \times \frac{5}{4}} = \frac{23}{9} \times \frac{\sqrt{5}}{2} = \frac{23\sqrt{5}}{18} \text{ units.}$$

16. Find the direction in which a straight line must be drawn through the point (– 1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

Sol. Let the required line through P(– 1, 2) meet the line $x + y = 4$ at (h, k). Then (h, k) must satisfy it.

$\therefore \quad h + k = 4 \quad ...(i)$

Distance between (– 1, 2) and (h, k) is 3 (given)



$\therefore \quad \sqrt{(h+1)^2 + (k-2)^2} = 3$

Putting $k = 4 - h$ from (*i*),

$\sqrt{(h+1)^2 + (4-h-2)^2} = 3$

$(h + 1)^2 + (2 - h)^2 = 9$

$\Rightarrow h^2 + 1 + 2h + 4 + h^2 - 4h = 9$

$\Rightarrow 2h^2 - 2h - 4 = 0$

$\Rightarrow$ Dividing by 2, $h^2 - h - 2 = 0 \Rightarrow (h - 2)(h + 1) = 0$

$\Rightarrow \quad h = 2 \text{ or } -1$

When $h = 2, k = 4 - h = 4 - 2 = 2$

When $h = -1, k = 4 - h = 4 - (-1) = 5$

$\therefore$ The points of intersection of lines through P (–1, 2) with the line $x + y = 4$ at a distance of 3 units from P are Q(2, 2) and R(– 1, 5).

Slope of PQ = $\dfrac{2-2}{2-(-1)} = 0$

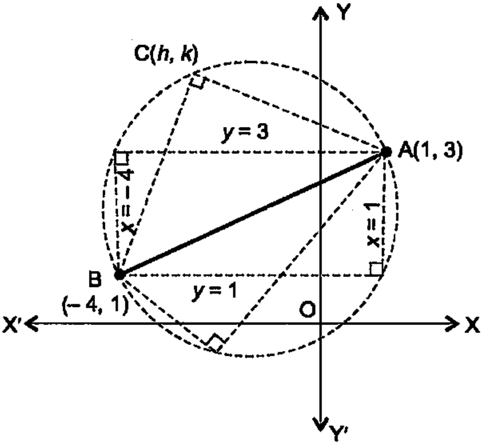
$\Rightarrow$ PQ is parallel to x-axis.

Slope of PR = $\dfrac{5-2}{-1-(-1)} = \dfrac{3}{0} = \infty$ which is not defined.

$\Rightarrow$ PR is parallel to y-axis.

Hence, the required line is parallel to x-axis or parallel to y-axis.

17. The hypotenuse of a right-angled triangle has its ends at the points (1, 3) and (– 4, 1). Find the equations of the legs (perpendicular sides) of the triangle.



Sol. The ends of the hypotenuse are A(1, 3) and B(– 4, 1). Let C(h, k) be the third vertex, then $\angle$ACB = 90°.

Slope of leg AC = $\dfrac{k-3}{h-1}$ ...(i)

Slope of leg BC = $\dfrac{k-1}{h+4}$ ...(ii)

Since AC $\perp$ BC, ($\because$ <ACB = 90°)

$$\therefore \left(\frac{k-3}{h-1}\right)\left(\frac{k-1}{h+4}\right) = -1$$

[$\because$ For perpendicularity $m_1m_2 = -1$]

$\Rightarrow$ $(k-3)(k-1) = -(h-1)(h+4)$

$\Rightarrow$ $(h-1)(h+4) + (k-3)(k-1) = 0$ ...(iii)

This equation (iii) is satisfied by infinite number of values of h and k.

Putting $h = 1$ and $k = 1$ in (iii),

$(1-1)(1+4) + (1-3)(1-1) = 0$ i.e., $0 + 0 = 0$ which is true.

$\therefore$ $h = 1$ and $k = 1$ satisfy (iii)

$\therefore$ co-ordinates of vertex C are $(h, k) = (1, 1)$

Putting $h = 1$, $k = 1$ *in* (i), slope of leg AC = $\dfrac{1-3}{1-1} = -\dfrac{2}{0}$

$= -\infty$, undefined.

∴ Leg AC is parallel to y-axis.

Hence equation of leg AC through A(1, 3) = (x_1, y_1) is $x = x_1$ *i.e.*, $x = 1$

Putting $h = 1, k = 1$ in (*ii*), slope of leg BC = $\frac{1-1}{1+4} = 0$

∴ Leg BC is horizontal.

Hence equation of leg BC through B (–4, 1) = (x_1, y_1) is *i.e.*, $y = 1$

Remark. In fact C is any point on the circle drawn on AB as diameter. C is not unique. In any position, the coordinates of C must satisfy (*iii*) and hence legs AC and BC have infinite positions and hence infinite equations.

18. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

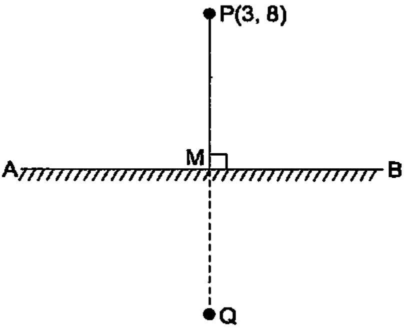
Sol. The given line is

AB : $x + 3y - 7 = 0$ ...(*i*)

The given point is P(3, 8).

Draw PM ⊥ AB and produce it to Q such that MQ = PM. Then Q is the image of P in the line AB.

(The image is as far behind the mirror as the object is in front of it.)



Slope of line (*i*) is $-\frac{a}{b} = \frac{-1}{3}$

∴ Slope of line PM perpendicular to AB is 3 (negative reciprocal)

∴ Equation of line PM is (point-slope form)

$y - 8 = 3(x - 3)$ or $y - 8 = 3x - 9$

or $-3x + y + 1 = 0$ or $3x - y - 1 = 0$ ...(*ii*)

To find foot of perpendicular M, the point of intersection of lines (*i*) and (*ii*), let us solve (*i*) and (*ii*) for x and y.

Eqn(*i*) + 3 × Eqn(*ii*) gives

$x + 3y - 7 + 9x - 3y - 3 = 0$ or $10x - 10 = 0$

or $10x = 10 \Rightarrow x = \frac{10}{10} = 1$

Putting $x = 1$ in (*i*), $1 + 3y - 7 = 0$ or $3y = 6$ *i.e.*, $y = 2$

∴ Foot of perpendicular M is (1, 2)

Let the co-ordinates of the image point Q be (α, β).

By definition of image point, M is mid-point of PQ

$\therefore \quad \frac{\alpha+3}{2} = 1$ and $\frac{\beta+8}{2} = 2 \quad \therefore \alpha + 3 = 2$ and $\beta + 8 = 4$.

$\Rightarrow \quad \alpha = -1$ and $\beta = -4$.

Hence, image point Q is $\leftrightarrow (-1, -4)$.

19. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

Sol. The given lines are

$$y = 3x + 1 \quad \text{...(i)}$$

$$2y = x + 3 \quad \text{or} \quad y = \frac{1}{2}x + \frac{3}{2} \quad \text{...(ii)}$$

and $$y = mx + c \quad \text{...(iii)}$$

All three equations (*i*), (*ii*) and (*iii*) are in slope-intercept form. Therefore their slopes are coefficiants of x in them.

Their slopes are 3, $\frac{1}{2}$ and m respectively.

Let the lines (*i*) and (*ii*) both make same angle θ with line (*iii*).

[$\because$ given: both (*i*) and (*ii*) are equally inclined to (*iii*)]

Then, $$\tan\theta = \left|\frac{m-3}{1+m\cdot 3}\right| = \left|\frac{m-\frac{1}{2}}{1+m\cdot\frac{1}{2}}\right|$$

$$\Rightarrow \quad \left|\frac{m-3}{1+3m}\right| = \left|\frac{2m-1}{2+m}\right| \Rightarrow \frac{m-3}{1+3m} = \pm\frac{2m-1}{2+m}$$

[$\because |x| = |y| \Rightarrow x = \pm y$]

cross-multiplying $(m - 3)(2 + m) = \pm(1 + 3m)(2m - 1)$

Taking + ve sign

$$(m-3)(2+m) = (1+3m)(2m-1)$$

$$\Rightarrow \quad m^2 - m - 6 = 6m^2 - m - 1$$

$$\Rightarrow \quad -5m^2 = 5$$

$\Rightarrow \quad m^2 = -1 < 0$ which is not possible.

Taking – ve sign

$$(m-3)(2+m) = -(1+3m)(2m-1)$$

$$\Rightarrow \quad m^2 - m - 6 = -6m^2 + m + 1$$

$$\Rightarrow \quad 7m^2 - 2m - 7 = 0$$

$$\Rightarrow \quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4+196}}{14} = \frac{2 \pm \sqrt{200}}{14}$$

$$= \frac{2 \pm 10\sqrt{2}}{14} = \frac{1 \pm 5\sqrt{2}}{7}.$$

20. If sum of the perpendicular distances of a variable point P(x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10, show that P must move on a line.

Sol. Given lines are $x + y - 5 = 0$ ...(i)

and $3x - 2y + 7 = 0$ ...(ii)

Since, distance of P(x, y) from (i) + distance of P(x, y) from (ii) = 10 (given)

$$\therefore \quad \frac{|x+y-5|}{\sqrt{1^2+1^2}} + \frac{|3x-2y+7|}{\sqrt{3^2+(-2)^2}} = 10$$

$$\Rightarrow \quad \pm\frac{x+y-5}{\sqrt{2}} \pm \frac{3x-2y+7}{\sqrt{13}} = 10$$

$$\Rightarrow \quad \pm\sqrt{13}(x + y - 5) \pm \sqrt{2}(3x - 2y + 7) = 10\sqrt{26}$$

On arrangement of terms, the above equations reduce to the form $Ax + By + C = 0$. This is a first degree equation in x and y and therefore, represents a straight line.

Hence, P(x, y) moves on a line.

21. Find equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

Sol. Given parallel lines are $9x + 6y - 7 = 0$ ...(i)

and $3x + 2y + 6 = 0$

Multiplying this equation by 3 so as to have same terms of x and y as in (i)

$$9x + 6y + 18 = 0 \quad ...(ii)$$

The line equidistant from (i) and (ii) must be parallel to them.

Let its equation be $9x + 6y + k = 0$ ...(iii)

Since (iii) is equidistant from (i) and (ii), therefore, distance between parallel lines (i) and (iii) = distance between parallel lines (ii) and (iii)

$$\Rightarrow \quad \frac{|k-(-7)|}{\sqrt{9^2+6^2}} = \frac{|k-18|}{\sqrt{9^2+6^2}} \qquad \left[d = \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}\right.$$

$$\Rightarrow \quad |k + 7| = |k - 18|$$

$$\Rightarrow \quad k + 7 = \pm(k - 18) \qquad [|x| = |y| \Rightarrow x = \pm y]$$

Taking + ve sign

$$k + 7 = k - 18$$

or $7 = -18$ which is false.

Taking – ve sign

$$k + 7 = -(k - 18)$$

or $\quad k + 7 = -k + 18 \quad$ or $\quad 2k = 11 \quad \therefore k = \dfrac{11}{2}$

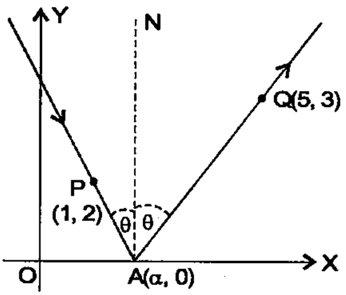
Putting this value of k in (*iii*), the equation of the required line is

$$9x + 6y + \frac{11}{2} = 0$$

or $\quad 18x + 12y + 11 = 0.$

22. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

Sol. Let the coordinates of point A on x-axis be (α, 0). Let the incident ray PA where point P is (1, 2) make an angle θ with the normal AN, then the reflected ray AQ where point Q is (5, 3) also makes angle θ with normal AN.



(By Law of Reflection)

Since $\quad \angle XAQ = 90° - \theta$

and $\quad \angle XAP = 90° + \theta$

$\therefore$ Slope of AQ = tan (90° − θ) = cot θ

Slope of AP = (tan 90° + θ) = − cot θ

$\Rightarrow$ Slope of AQ + Slope of AP = 0

$$\Rightarrow \quad \frac{3-0}{5-a} + \frac{2-0}{1-a} = 0 \quad \Rightarrow \quad \frac{3(1-a)+2(5-a)}{(5-a)(1-a)} = 0$$

$$\Rightarrow \quad 3 - 3a + 10 - 2a = 0 \quad \Rightarrow \quad -5a = -13$$

$$\therefore \quad a = \frac{13}{5}$$

Hence, the coordinates of A are $(a, 0) = \left(\dfrac{13}{5}, 0\right)$.

23. Prove that the product of the lengths of the perpendiculars drawn from the points $(\sqrt{a^2-b^2}, 0)$ and $(-\sqrt{a^2-b^2}, 0)$ to the line $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$ is b^2.

Sol. Given line is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0$ ...(i)

p_1 = length of perpendicular from $(\sqrt{a^2-b^2}, 0)$ on (i)

$$\left(\text{using the formula } \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}\right)$$

$$= \frac{\left|\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 0 - 1\right|}{\sqrt{\left(\frac{\cos\theta}{a}\right)^2+\left(\frac{\sin\theta}{b}\right)^2}} = \frac{\left|\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

p_2 = length of perpendicular from $(-\sqrt{a^2-b^2}, 0)$ on (i)

$$= \frac{\left|\frac{-\sqrt{a^2-b^2}}{a}\cos\theta + 0 - 1\right|}{\sqrt{\left(\frac{\cos\theta}{a}\right)^2+\left(\frac{\sin\theta}{b}\right)^2}} = \frac{\left|-\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1\right)\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$= \frac{\left|\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} \quad [\because |-x| = |x|]$$

$$\therefore \quad p_1 p_2 = \frac{\left|\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}} \times \frac{\left|\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1\right|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$= \frac{\left|\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right)\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1\right)\right|}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}$$

$[\because |x||y| = |xy|]$

$$= \frac{\left|\frac{a^2-b^2}{a^2}\cos^2\theta - 1\right|}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}} = \frac{\left|\frac{(a^2-b^2)\cos^2\theta - a^2}{a^2}\right|}{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}}$$

$$= \frac{|a^2\cos^2\theta - b^2\cos^2\theta - a^2|}{a^2} \times \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{b^2|-a^2(1-\cos^2\theta) - b^2\cos^2\theta|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{b^2|-a^2\sin^2\theta - b^2\cos^2\theta|}{b^2\cos^2\theta + a^2\sin^2\theta} = \frac{b^2|-(b^2\cos^2\theta + a^2\sin^2\theta)|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

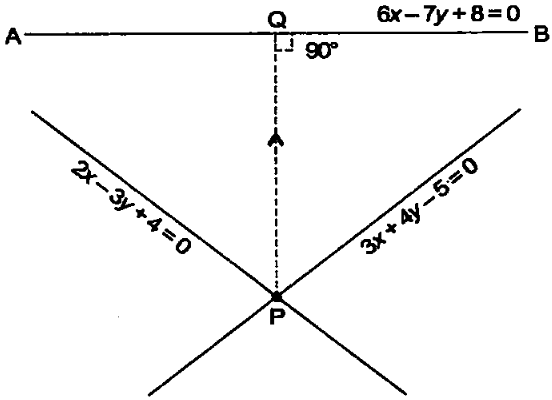
$$= \frac{b^2|b^2\cos^2\theta + a^2\sin^2\theta|}{b^2\cos^2\theta + a^2\sin^2\theta} \qquad [\because \ |-x| = |x|]$$

$$= \frac{b^2(b^2\cos^2\theta + a^2\sin^2\theta)}{b^2\cos^2\theta + a^2\sin^2\theta} = b^2. \qquad [\because \ |x| = x \text{ if } x > 0]$$

Hence, $p_1p_2 = b^2$.

24. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Sol.



The equations of two straight paths are

$2x - 3y + 4 = 0$ ...(*i*)

and $3x + 4y - 5 = 0$ ...(*ii*)

Let us solve eqn (*i*) and (*ii*) for x and y to find co-ordinates of the point P [Junction or crossing of paths (*i*) and (*ii*)]

Eqn (*i*) × 4 + Eqn (*ii*) × 3 gives to eliminate y.

$8x - 12y + 16 + 9x + 12y - 15 = 0$

or $17x + 1 = 0$ or $17x = -1 \Rightarrow x = \frac{-1}{17}$

Putting $x = \frac{-1}{17}$ in (*i*), $-\frac{2}{17} - 3y + 4 = 0$

$$\Rightarrow -3y = -4 + \frac{2}{17} = \frac{-68+2}{17} = \frac{-66}{17}$$

$$\Rightarrow y = \frac{66}{3\times17} = \frac{22}{17}$$

∴ The person is standing at junction P $\left(-\frac{1}{17}, \frac{22}{17}\right)$.

Equation of the path AB he wants to reach is

$6x - 7y + 8 = 0$ ...(*iii*)

For least time, the person must follow the path PQ where PQ ⊥ AB.

(The shortest distance of a point from a line is the perpendicular distance).

From (*iii*), slope of AB $= -\frac{a}{b} = -\frac{6}{-7} = \frac{6}{7}$.

Since PQ ⊥ AB, therefore, slope of PQ $= -\frac{7}{6}$. | negative reciprocal

∴ Equation of PQ, the line through P$\left(-\frac{1}{17}, \frac{22}{17}\right)$ having slope $-\frac{7}{6}$ is

$$y - \frac{22}{17} = -\frac{7}{6}\left(x + \frac{1}{17}\right)$$ [Point-slope Form]

$$\Rightarrow \quad 6y - \frac{132}{17} = -7x - \frac{7}{17}$$

$$\Rightarrow \quad 7x + 6y = \frac{125}{17}$$

cross-multiplying $119x + 102y = 125$.