

# 4

# Principle of Mathematical Induction



## Lesson at a Glance

**1. Deduction:** It is the application of a general case (or result) to the particular case (or situation).

**2. Induction:** It means generalisation from particular cases or facts.

### 3. Principle of Mathematical Induction

To establish the truth of statements formulated in terms of  $n$ , where  $n$  is a positive integer, we use **the principle of mathematical induction** stated below: Suppose there is a given statement  $P(n)$  involving the natural numbers  $n$  such that

(i) the statement is true for  $n = 1$  i.e.,  $P(1)$  is true and

(ii) the truth of  $P(k)$ , where  $k$  is some positive integer, implies the truth of  $P(k + 1)$  i.e., **if the statement is true for  $n = k$  (where  $k$  is some positive integer), then the statement is also true for  $n = k + 1$ .**

Then,  $P(n)$  is true for all natural numbers  $n$ .

## TEXTBOOK QUESTIONS SOLVED

### EXERCISE 4.1 (Page No.: 94-95)

**Prove the following by using the principle of mathematical induction for all  $n \in \mathbb{N}$ .**

$$1. 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}.$$

**Sol.** Let  $P(n) : 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$ . ... (i)

**Step I:** Putting  $n = 1$  in (i)

$$P(1) : 1 = \frac{3^1 - 1}{2} \quad \text{or} \quad 1 = 1 \text{ which is true.}$$

Thus  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$

$$\text{i.e., let } 1 + 3 + 3^2 + \dots + 3^{k-1} = \frac{3^k - 1}{2} \quad \dots(ii)$$

**Step III:** We want to prove that [changing  $k$  to  $k + 1$  in (ii)]

$P(k + 1) : 1 + 3 + 3^2 + \dots + 3^{k-1} + 3^k = \frac{3^{k+1} - 1}{2}$  is also true.

Now, L.H.S. of  $P(k + 1)$

$$= 1 + 3 + 3^2 + \dots + 3^{k-1} + 3^k = \frac{3^k - 1}{2} + 3^k \quad [\text{Using (ii)}]$$

$$= \frac{3^k - 1 + 2 \cdot 3^k}{2} = \frac{(1+2)3^k - 1}{2} = \frac{3 \cdot 3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2} = \text{R.H.S. of } P(k + 1)$$

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true.

Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$2. 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2.$$

$$\text{Sol. Let } P(n) : 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \quad \dots(i)$$

**Step I:** Putting  $n = 1$  in (i),  $P(1) : 1^3 = \left[ \frac{1(1+1)}{2} \right]^2$  or

$1 = 1$  which is true.

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

$$\text{i.e., let } 1^3 + 2^3 + 3^3 + \dots + k^3 = \left[ \frac{k(k+1)}{2} \right]^2 \quad \dots(ii)$$

**Step III:** We want to prove that [changing  $k$  to  $k+1$  in (ii)]

$$P(k + 1) : 1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3$$

$$= \left[ \frac{(k+1)(k+2)}{2} \right]^2$$

is also true

Now, L.H.S. of  $P(k+1) = 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3$

$$= \left[ \frac{k(k+1)}{2} \right]^2 + (k+1)^3 \quad \text{[Using (ii)]}$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2 \left[ \frac{k^2}{4} + (k+1) \right]$$

$$= (k+1)^2 \left[ \frac{k^2 + 4k + 4}{4} \right] = (k+1)^2 \frac{(k+2)^2}{4}$$

$$\begin{aligned} [\because k^2 + 4k + 4 &= k^2 + 2k + 2k + 4 = k(k+2) + 2(k+2)] \\ &= (k+2)(k+2) = (k+2)^2] \end{aligned}$$

$$= \left[ \frac{(k+1)(k+2)}{2} \right]^2 = \text{R.H.S. of } P(k+1)$$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true.

Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$3. \quad 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+n} = \frac{2n}{n+1}.$$

$$\text{Sol. Let } P(n) : 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+n}$$

$$= \frac{2n}{n+1}$$

**Step I:** Putting  $n = 1$  in (i), ...(i)

$$P(1) : 1 = \frac{2 \times 1}{1+1} \quad \text{or} \quad 1 = 1 \quad \text{which is true.}$$

Thus,  $P(1)$  is true

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

$$\text{i.e., let } 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k}$$

$$= \frac{2k}{k+1} \quad \text{...(ii)}$$

**Step III:** We want to prove that [changing  $k$  to  $k+1$  in (ii)]

$$P(k + 1) : 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} + \frac{1}{1+2+3+\dots+k+(k+1)} = \frac{2(k+1)}{k+2}$$

is also true.

Now, L.H.S. of  $P(k+1) = 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots +$

$$\frac{1}{1+2+3+\dots+k} + \frac{1}{1+2+3+\dots+k+(k+1)}$$

$$= \frac{2k}{k+1} + \frac{1}{1+2+3+\dots+k+(k+1)} \quad \text{[Using (ii)]}$$

$$= \frac{2k}{k+1} + \frac{1}{\frac{k+1}{2}(1+k+1)}$$

[ $\because 1+2+3+\dots+k+(k+1)$  is an A.P. of  $(k+1)$  terms

$$\therefore S_n = \frac{n}{2}(a+l) \Rightarrow S_{k+1} = \frac{k+1}{2}(1+k+1)]$$

$$= \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)} = \frac{2}{k+1} \left( k + \frac{1}{k+2} \right)$$

$$= \frac{2}{k+1} \left( \frac{k^2 + 2k + 1}{k+2} \right) = \frac{2}{k+1} \cdot \frac{(k+1)^2}{k+2}$$

$$= \frac{2(k+1)}{k+2} = \text{R.H.S. of } P(k+1)$$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true.

Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

4.  $1.2.3 + 2.3.4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ .

**Sol.** Let  $P(n)$  be the statement

$$"1.2.3 + 2.3.4 + \dots + n(n+1)(n+2)"$$

$$= \frac{n(n+1)(n+2)(n+3)}{4} \quad \dots(i)$$

**Step I:** Then  $P(1)$  is the statement, [Putting  $n = 1$  in (i)]

$$"1.2.3 = \frac{1 \times 2 \times 3 \times 4}{4}" \text{ which is true.}$$

$\therefore P(1)$  is true

**Step II:** Suppose  $P(k)$  is true, [Putting  $n = k$  in (i)]

i.e., let  $1.2.3 + 2.3.4 + \dots + k(k+1)(k+2)$

$$= \frac{k(k+1)(k+2)(k+3)}{4} \quad \dots(ii)$$

**Step III:**  $P(k+1)$  is the statement, [Putting  $n = k+1$  in (i)]

" $1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}."$$
 We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of  $P(k+1)$ , we have

$$1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

$$= \frac{(k+1)(k+2)(k+3)}{4} (k+4) = \text{R.H.S. of } P(k+1)$$

$$\Rightarrow P(k+1) \text{ is true.}$$

$\therefore$  By the principle of mathematical induction,  $P(n)$  is true for all natural numbers  $n$ .

$$5. \quad 1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}.$$

**Sol.** Let  $P(n) : 1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n$

$$= \frac{(2n-1)3^{n+1} + 3}{4} \quad \dots(i)$$

**Step I:** Then  $P(1)$  is the statement putting  $n = 1$  in (i)

$$P(1) : 1.3 = \frac{(2 \times 1 - 1)3^{1+1} + 3}{4} \quad \text{or} \quad 3 = \frac{9 + 3}{4}$$

or  $3 = 3$  which is true. Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

$$\text{i.e., let } 1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k = \frac{(2k-1)3^{k+1} + 3}{4} \quad \dots(ii)$$

**Step III:** We want to prove that [changing  $k$  to  $k + 1$  in (ii)]

$$P(k + 1) : 1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k + (k + 1).3^{k+1}$$

$$\begin{aligned} &= \frac{[2(k+1)-1].3^{k+2} + 3}{4} \\ &= \frac{(2k+1).3^{k+2} + 3}{4} \text{ is also true.} \end{aligned}$$

$$\text{Now, L.H.S. of } P(k + 1) = 1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k + (k + 1).3^{k+1}$$

$$\begin{aligned} &= \frac{(2k-1).3^{k+1} + 3}{4} + (k+1).3^{k+1} \quad [\text{Using (ii)}] \\ &= \frac{(2k-1).3^{k+1} + 3 + 4(k+1).3^{k+1}}{4} \\ &= \frac{(2k-1+4k+4).3^{k+1} + 3}{4} \\ &= \frac{(6k+3).3^{k+1} + 3}{4} = \frac{3(2k+1).3^{k+1} + 3}{4} \\ &= \frac{(2k+1).3^{k+2} + 3}{4} = \text{R.H.S. of } P(k + 1) \end{aligned}$$

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true.

Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$6. \quad 1.2 + 2.3 + 3.4 + \dots + n(n + 1) = \left[ \frac{n(n+1)(n+2)}{3} \right].$$

$$\text{Sol. Let } P(n) : 1.2 + 2.3 + 3.4 + \dots + n(n + 1)$$

$$= \frac{n(n+1)(n+2)}{3} \quad \dots(i)$$

**Step I:** Then  $P(1)$  is the statement [Putting  $n = 1$  in (i)]

$$P(1) : 1.2 = \frac{1 \times 2 \times 3}{3} \quad \text{or} \quad 2 = 2 \text{ which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

$$\text{i.e., let } 1.2 + 2.3 + 3.4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3} \dots(ii)$$

We want to prove that [changing  $k$  to  $(k+1)$  in (ii)]

$$P(k+1) : 1.2 + 2.3 + 3.4 + \dots + k(k+1) + (k+1)(k+2) \\ = \frac{(k+1)(k+2)(k+3)}{3} \text{ is also true.}$$

$$\text{Now, L.H.S. of } P(k+1) = 1.2 + 2.3 + 3.4 + \dots + k(k+1) \\ + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad [\text{Using (ii)}]$$

$$= (k+1)(k+2) \left( \frac{k}{3} + 1 \right)$$

$$= \frac{(k+1)(k+2)(k+3)}{3} = \text{R.H.S. of } P(k+1)$$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$7. 1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$$

**Sol.** Let  $P(n)$  be the statement

$$"1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}" \dots(i)$$

**Step I:** Then  $P(1)$  is the statement [Putting  $n = 1$  in (i)]

$$"1.3 = \frac{1 \times (4 \cdot 1^2 + 6 \cdot 1 - 1)}{3}, \text{ i.e., } 3 = 3" \text{ which is true.}$$

$\therefore P(1)$  is true.

**Step II:** Suppose  $P(k)$  is true, [Putting  $n = k$  in (i)]

$$\text{i.e., let } 1.3 + 3.5 + \dots + (2k-1)(2k+1) = \frac{k(4k^2+6k-1)}{3} \dots(ii)$$

**Step III:** Putting  $n = k+1$  in (i):

$P(k+1)$  is the statement

$$"1.3 + 3.5 + \dots + (2k-1)(2k+1) + (2k+1)(2k+3)"$$

$$= \frac{(k+1)[4(k+1)^2 + 6(k+1) - 1]}{3}$$

$$= \frac{(k+1)(4k^2 + 14k + 9)}{3} \quad [\because (k+1)^2 = k^2 + 2k + 1]$$

We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of  $P(k + 1)$ , we have  $1.3 + 3.5 + \dots + (2k - 1)(2k + 1) + (2k + 1)(2k + 3)$

$$\begin{aligned} &= \frac{k(4k^2 + 6k - 1)}{3} + (2k + 1)(2k + 3) \\ &= \frac{1}{3} [(4k^3 + 6k^2 - k) + 3(4k^2 + 8k + 3)] \\ &= \frac{1}{3} (4k^3 + 18k^2 + 23k + 9) \qquad \dots(iii) \end{aligned}$$

Breaking terms to get  $(k + 1)$  as a factor as required in R.H.S. of  $P(k + 1)$

$$\begin{aligned} &= \frac{1}{3} (4k^3 + 4k^2 + 14k^2 + 14k + 9k + 9) \text{ Note} \\ &= \frac{1}{3} [4k^2(k + 1) + 14k(k + 1) + 9(k + 1)] \\ &= \frac{(k + 1)(4k^2 + 14k + 9)}{3} \\ &= \frac{(k + 1)[4k^2 + 8k + 4 + 6k + 6 - 1]}{3} \\ &= \frac{(k + 1)[4(k^2 + 2k + 1) + 6(k + 1) - 1]}{3} \\ &= \frac{(k + 1)[4(k + 1)^2 + 6(k + 1) - 1]}{3} \end{aligned}$$

$\Rightarrow P(k + 1)$  is true.

$\therefore$  By the principle of mathematical induction,  $P(n)$  is true for all natural numbers  $n$ .

**Remark:** After arriving at Eqn (iii) in the above solution, we can also proceed as below.

$$\begin{aligned} \text{R.H.S. of } P(k + 1) &= \frac{(k + 1)[4k^2 + 14k + 9]}{3} \\ &= \frac{1}{3} (4k^3 + 14k^2 + 9k + 4k^2 + 14k + 9) \\ &= \frac{1}{3} (4k^3 + 18k^2 + 23k + 9) \qquad \dots(iv) \end{aligned}$$

From (iii) and (iv) L.H.S. of  $P(k + 1) = \text{R.H.S. of } P(k + 1)$



$\therefore P(k + 1)$  is true.

$\therefore$  By the principle of mathematical induction,  $P(n)$  is true for all natural numbers  $n$ .

$$8. 1.2 + 2.2^2 + 3.2^3 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$$

**Sol.** Let  $P(n) : 1.2 + 2.2^2 + 3.2^3 + \dots + n.2^n = (n - 1) 2^{n+1} + 2 \dots (i)$

**Step I:** Therefore  $P(1)$  is the statement [Putting  $n = 1$  in (i)]

$P(1) : 1.2 = 0 + 2$  or  $2 = 2$  which is true.

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

*i.e.*, let  $1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k = (k - 1).2^{k+1} + 2 \dots (ii)$

**Step III:** We want to prove that

$P(k + 1) : 1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k + (k + 1).2^{k+1}$   
 $= k.2^{k+2} + 2$  is also true.

Now, L.H.S. of  $P(k + 1) = 1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k + (k + 1).2^{k+1}$

$$= (k - 1).2^{k+1} + 2 + (k + 1).2^{k+1} \quad [\text{Using (ii)}]$$

$$= (k - 1 + k + 1).2^{k+1} + 2$$

$$= 2k.2^{k+1} + 2 = k.2^{k+2} + 2 = \text{R.H.S. of } P(k + 1)$$

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$9. \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

**Sol.** Let  $P(n)$  be the statement

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \dots (i)$$

**Step I:** Putting  $n = 1$  in (i),

$P(1)$  is the statement " $\frac{1}{2} = 1 - \frac{1}{2^1}$ " which is true.

$\therefore P(1)$  is true.

**Step II:** Suppose  $P(k)$  is true, [Putting  $n = k$  in (i)]

*i.e.*, let

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \quad \dots(ii)$$

**Step III.** Putting  $n = k + 1$  in (i),  $P(k + 1)$  is the statement

$$“ \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}} ”.$$

We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of  $P(k + 1)$ , we have

$$\begin{aligned} \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= \left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{(2^k)} + \frac{1}{(2^k 2^1)} \\ &= 1 - \frac{1}{(2^k)} \left(1 - \frac{1}{2}\right) = 1 - \frac{1}{(2^k)} \cdot \frac{1}{2} \\ &= 1 - \frac{1}{(2^{k+1})} = \text{R.H.S. of } P(k + 1) \end{aligned}$$

$\Rightarrow P(k + 1)$  is true.

$\therefore$  By the principle of mathematical induction,  $P(n)$  is true for all natural numbers  $n$ .

$$10. \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}.$$

$$\text{Sol. Let } P(n) : \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4} \quad \dots(i)$$

**Step I:** Therefore  $P(1)$  is the statement [Putting  $n=1$  in...(i)]

$$P(1) : \frac{1}{2.5} = \frac{1}{6 \times 1 + 4} \quad \text{or} \quad \frac{1}{10} = \frac{1}{10} \quad \text{which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

$$\text{i.e., let } \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \quad \dots(ii)$$

**Step III:** Changing  $k$  to  $(k+1)$  in (ii),  $P(k+1)$  is the statement.

$$P(k+1) : \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} \\ + \frac{1}{(3k+2)(3k+5)} = \frac{k+1}{6k+10}$$

Let us prove that  $P(k+1)$  is true.

$$\begin{aligned} \text{Now, L.H.S. of } P(k+1) &= \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \\ &\frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)} \\ &= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} \quad [\text{Using (ii)}] \\ &= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)} \\ &= \frac{1}{3k+2} \left[ \frac{k}{2} + \frac{1}{3k+5} \right] = \frac{1}{3k+2} \cdot \frac{k(3k+5)+2}{2(3k+5)} \\ &= \frac{3k^2+5k+2}{(3k+2)(6k+10)} = \frac{3k^2+3k+2k+2}{(3k+2)(6k+10)} \\ &= \frac{(k+1)(3k+2)}{(3k+2)(6k+10)} \\ &= \frac{k+1}{6k+10} = \text{R.H.S. of } P(k+1) \end{aligned}$$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$11. \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} \\ = \frac{n(n+3)}{4(n+1)(n+2)}$$

$$\text{Sol. Let } P(n) : \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} \\ = \frac{n(n+3)}{4(n+1)(n+2)} \quad \dots(i)$$

**Step I:** Therefore  $P(1)$  is the statement [Putting  $n=1$  in (i)]

$$P(1) : \frac{1}{1.2.3} = \frac{1 \times 4}{4 \times 2 \times 3} \quad \text{or} \quad \frac{1}{6} = \frac{1}{6} \quad \text{which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$  i.e., let

$$\begin{aligned} & \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{k(k+1)(k+2)} \\ &= \frac{k(k+3)}{4(k+1)(k+2)} \quad \dots(ii) \end{aligned}$$

**Step III:** We want to prove that [changing  $k$  to  $(k+1)$  in (ii)]

$$\begin{aligned} P(k+1) : & \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{k(k+1)(k+2)} \\ &+ \frac{1}{(k+1)(k+2)(k+3)} \\ &= \frac{(k+1)(k+4)}{4(k+2)(k+3)} \end{aligned}$$

is also true.

$$\begin{aligned} \text{Now, L.H.S of } P(k+1) &= \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \\ & \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\ &= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad [\text{Using (ii)}] \\ &= \frac{1}{(k+1)(k+2)} \left[ \frac{k(k+3)}{4} + \frac{1}{k+3} \right] \\ &= \frac{1}{(k+1)(k+2)} \cdot \frac{k(k+3)^2 + 4}{4(k+3)} \\ &= \frac{k(k^2 + 6k + 9) + 4}{4(k+1)(k+2)(k+3)} \end{aligned}$$

$$= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \quad \dots(iii)$$

Splitting Numerator to get  $(k + 1)$  as a factor which is present in the Denominator here but absent in the Denominator of required R.H.S.

$$= \frac{k^3 + k^2 + 5k^2 + 5k + 4k + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^2(k+1) + 5k(k+1) + 4(k+1)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k^2 + 5k + 4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} \quad [\because k^2 + 5k + 4 = (k+1)(k+4)]$$

$$= \text{R.H.S. of P}(k+1)$$

$\Rightarrow$   $P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**Remark:** After arriving at Eqn (iii) in the above solution, we can also proceed as below:

$$\text{R.H.S. of P}(k+1) = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

Multiplying both Numerator and denominator by  $(k+1)$  which is present as a factor in the denominator of (iii) but is absent in the denominator of the R.H.S. (above).

$$= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)} = \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k^2 + 2k + 1)(k+4)}{4(k+1)(k+2)(k+3)} = \frac{k^3 + 4k^2 + 2k^2 + 8k + k + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \quad \dots(iv)$$

From (iii) and (iv) L.H.S. of  $P(k+1) = \text{R.H.S. of } P(k+1)$

$\therefore P(k+1)$  is true when  $P(k)$  is true.

Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$12. a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

**Sol.** Let  $P(n) : a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$  ... (i)

**Step I:** Therefore  $P(1)$  is the statement [Putting  $n=1$  in (i)]

$$P(1) = a = \frac{a(r-1)}{r-1} \quad \text{or} \quad a = a \quad \text{which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

i.e., let  $a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1}$  ... (ii)

**Step III** We want to prove that [changing  $k$  to  $k + 1$  in (ii)]

$$\begin{aligned} P(k+1) : a + ar + ar^2 + \dots + ar^{k-1} + ar^k \\ = \frac{a(r^{k+1} - 1)}{r - 1} \end{aligned}$$

is also true.

Now, L.H.S. of  $P(k+1) = a + ar + ar^2 + \dots + ar^{k-1} + ar^k$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k \quad \text{[Using (ii)]}$$

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

$$= \frac{a(r^k - 1 + r^{k+1} - r^k)}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1} = \text{R.H.S. of } P(k+1)$$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$13. \left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2.$$

**Sol.** Let  $P(n)$  be the statement

$$\left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2n+1}{n^2}\right) = (n+1)^2 \quad \dots(i)$$

**Step I:** Putting  $n = 1$  in (i),

$$P(1) \text{ is the statement } \left(1 + \frac{3}{1}\right) = (1+1)^2$$

or  $4 = 2^2$  or  $4 = 4$  which is true.

Thus  $P(1)$  is true.

**Step II:** Let  $P(k)$  be true i.e.,  $P(n)$  be true for  $n = k$ .

Putting  $n = k$  in (i), we have

$$\left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) = (k+1)^2 \quad \dots(ii)$$

**Step III:** Putting  $n = k + 1$  in (i), [on changing  $k$  to  $(k + 1)$  in (ii)]  $P(k + 1)$  is the statement

$$\begin{aligned} \left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) \left[1 + \frac{2(k+1)+1}{(k+1)^2}\right] \\ = (k+1+1)^2 \end{aligned}$$

$$\begin{aligned} \text{i.e., } \left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) \left(1 + \frac{2k+3}{(k+1)^2}\right) \\ = (k+2)^2 \end{aligned}$$

We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of  $P(k + 1)$ , we have

$$\left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{2k+1}{k^2}\right) \left(1 + \frac{2k+3}{(k+1)^2}\right)$$

$$\begin{aligned}
 &= (k + 1)^2 \left[ 1 + \frac{2k + 3}{(k + 1)^2} \right] \\
 &= (k + 1)^2 \left[ \frac{(k + 1)^2 + 2k + 3}{(k + 1)^2} \right] \\
 &= k^2 + 1 + 2k + 2k + 3 = k^2 + 4k + 4 = (k + 2)^2
 \end{aligned}$$

∴ P(k + 1) is true.

∴ P(n) is true for all natural numbers n by P.M.I.

14.  $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = n + 1.$

**Sol.** Let P(n) :  $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = n + 1 \dots(i)$

**Step I:** Therefore P (1) is the statement [Putting n = 1 in (i)]

P(1) :  $1 + \frac{1}{1} = 1 + 1$  or  $2 = 2$  which is true.

Thus, P(1) is true.

**Step II:** Assume that P(k) is true for some natural number k.

[Putting n = k in (i)]

i.e., let  $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right) = k + 1 \dots(ii)$

**Step III:** We want to prove that [changing k to k + 1 in (ii)]

P(k + 1) :  $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k + 1}\right) = k + 2$

is also true.

Now, L.H.S. of P (k + 1)

$= \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k + 1}\right)$

$= (k + 1)\left(1 + \frac{1}{k + 1}\right)$  [Using (ii)]



$$= (k + 1) \left( \frac{k + 1 + 1}{k + 1} \right) = k + 2$$

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$15. 1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$$

**Sol.** Let  $P(n)$  be the statement

$$"1^2 + 3^2 + 5^2 \dots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}." \quad \dots(i)$$

**Step I:** Then  $P(1)$  is the statement [Putting  $n = 1$  in (i)]

$$"1^2 = \frac{1 \times (2 \times 1 - 1)(2 \times 1 + 1)}{3}, \text{ i.e., } 1 = 1" \text{ which is true.}$$

$\therefore P(1)$  is true.

**Step II:** Suppose  $P(k)$  is true, [Putting  $n = k$  in (i)]

i.e., let

$$1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3} \quad \dots(ii)$$

**Step III:**  $P(k + 1)$  is the statement [Putting  $n = k + 1$  in (i)]

$$"1^2 + 3^2 + \dots + (2k - 1)^2 + (2k + 1)^2 = \frac{(k + 1)(2k + 1)(2k + 3)}{3}."$$

We have to prove it.

Putting the value of L.H.S. of eqn. (ii) in L.H.S. of  $P(k + 1)$ , we have

$$1^2 + 3^2 + \dots + (2k - 1)^2 + (2k + 1)^2 = \frac{k(2k - 1)(2k + 1)}{3} + (2k + 1)^2$$

Now taking  $(2k + 1)$  common from R.H.S. and taking 3 as L.C.M. on R.H.S.;

$$\begin{aligned} &= \frac{2k + 1}{3} [k(2k - 1) + 3(2k + 1)] \\ &= \frac{2k + 1}{3} (2k^2 + 5k + 3) \\ &= \frac{2k + 1}{3} (2k^2 + 2k + 3k + 3) \end{aligned}$$

$$\begin{aligned}
 &= \frac{2k+1}{3} [2k(k+1) + 3(k+1)] \\
 &= \frac{(k+1)(2k+1)(2k+3)}{3}
 \end{aligned}$$

$\Rightarrow P(k+1)$  is true.

$\therefore$  By the principle of mathematical induction,  $P(n)$  is true for all natural numbers  $n$ .

16. 
$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

Sol. Let  $P(n) : \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)}$   

$$= \frac{n}{3n+1} \quad \dots(i)$$

**Step I:** Therefore  $P(1)$  is the statement [Putting  $n = 1$  in (i)]

$$P(1) = \frac{1}{1.4} = \frac{1}{3 \times 1 + 1} \quad \text{or} \quad \frac{1}{4} = \frac{1}{4} \quad \text{which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ . [Putting  $n = k$  in (i)]

i.e., let 
$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)}$$
  

$$= \frac{k}{3k+1} \quad \dots(ii)$$

**Step III:** We want to prove that [changing  $k$  to  $k+1$  in (ii)]

$$\begin{aligned}
 P(k+1) : & \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots \\
 & + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}
 \end{aligned}$$

is also true.

$$\begin{aligned}
 \text{Now, L.H.S. of } P(k+1) &= \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \\
 &\frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} \\
 &= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \quad [\text{Using (i)}] \\
 &= \frac{1}{3k+1} \left[ k + \frac{1}{3k+4} \right] \quad \left\{ \begin{array}{l} \because 3k^2 + 4k + 1 \\ = 3k^2 + 3k + k + 1 \\ = 3k(k+1) + 1(k+1) \\ = (3k+1)(k+1) \end{array} \right. \\
 &= \frac{1}{3k+1} \cdot \frac{3k^2 + 4k + 1}{3k+4} \\
 &= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = \text{R.H.S. of } P(k+1)
 \end{aligned}$$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

$$17. \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}.$$

$$\begin{aligned}
 \text{Sol. Let } P(n) : \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} \\
 = \frac{n}{3(2n+3)} \quad \dots(i)
 \end{aligned}$$

**Step I:** Putting  $n = 1$  in (i),

$$P(1) : \frac{1}{3.5} = \frac{1}{3(2 \times 1 + 3)} \quad \text{or} \quad \frac{1}{15} = \frac{1}{15} \quad \text{which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

$$\begin{aligned}
 \text{i.e., let } \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} \\
 = \frac{k}{3(2k+3)} \quad \dots(ii)
 \end{aligned}$$

**Step III:** We want to prove that [changing  $k$  to  $(k+1)$  in (ii)]

$$P(k + 1) : \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2k+3)(2k+5)} = \frac{k+1}{3(2k+5)}$$

is also true.

$$\begin{aligned} \text{Now, L.H.S. of } P(k + 1) &= \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} + \frac{1}{(2k+3)(2k+5)} \\ &= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)} \quad \text{[Using (ii)]} \\ &= \frac{1}{2k+3} \left[ \frac{k}{3} + \frac{1}{2k+5} \right] \\ &= \frac{1}{2k+3} \cdot \frac{2k^2 + 5k + 3}{3(2k+5)} \quad \left| \begin{array}{l} \because 2k^2 + 5k + 3 \\ = 2k^2 + 2k + 3k + 3 \\ = 2k(k+1) + 3(k+1) \\ = (2k+3)(k+1) \end{array} \right. \\ &= \frac{1}{2k+3} \cdot \frac{(2k+3)(k+1)}{3(2k+5)} \\ &= \frac{k+1}{3(2k+5)} = \text{R.H.S. of } P(k + 1) \end{aligned}$$

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

18.  $1 + 2 + 3 + \dots + n < \frac{1}{8} (2n + 1)^2$ .

**Sol.** Let  $P(n) : 1 + 2 + 3 + \dots + n < \frac{1}{8} (2n + 1)^2$  ... (i)

**Step I:** Putting  $n = 1$  in (i),

$$P(1) : 1 < \frac{1}{8} (2 \times 1 + 1)^2 \text{ or } 1 < \frac{9}{8} \text{ which is true.}$$

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ . Putting  $n = k$  in (i)

i.e., let  $1 + 2 + 3 + \dots + k < \frac{1}{8} (2k + 1)^2$  ... (ii)

**Step III:** We want to prove that [changing  $k$  to  $(k + 1)$  in (ii)]

$$P(k + 1) : 1 + 2 + 3 + \dots + k + (k + 1) < \frac{1}{8} (2k + 3)^2$$

Now,

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k + 1) &< \frac{1}{8} (2k + 1)^2 + (k + 1) && \text{[Using (ii)]} \\ &= \frac{1}{8} [(2k + 1)^2 + 8(k + 1)] \\ &= \frac{1}{8} [4k^2 + 4k + 1 + 8k + 8] \\ &= \frac{1}{8} (4k^2 + 12k + 9) \\ &= \frac{1}{8} (2k + 3)^2 \end{aligned}$$

$$\begin{aligned} [\because 4k^2 + 12k + 9 &= 4k^2 + 6k + 6k + 9 \\ &= 2k(2k + 3) + 3(2k + 3) \\ &= (2k + 3)(2k + 3) = (2k + 3)^2] \end{aligned}$$

$$\therefore 1 + 2 + 3 + \dots + k + (k + 1) < \frac{1}{8} (2k + 3)^2$$

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**19.  $n(n + 1)(n + 5)$  is a multiple of 3.**

**Sol.** Let  $P(n) : n(n + 1)(n + 5)$  is a multiple of 3. ... (i)

**Step I:** Putting  $n = 1$  in (i),

$P(1) : 1(2)(6)$  is a multiple of 3 or 12 is a multiple of 3 which is true.

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

*i.e.,* let  $k(k + 1)(k + 5)$  be a multiple of 3

*i.e.,* let  $k(k^2 + 6k + 5)$  be a multiple of 3

*i.e.,*  $k^3 + 6k^2 + 5k = 3a$  where  $a \in \mathbb{Z}$

$$\Rightarrow k^3 = 3a - 6k^2 - 5k \quad \dots(ii)$$

**Step III:** We want to prove that

[putting  $n = k + 1$  in (i)]

$P(k + 1)$  :  $(k + 1)(k + 2)(k + 6)$  is a multiple of 3 is also true.

$$\begin{aligned} \text{Now, } (k + 1)(k + 2)(k + 6) &= (k + 1)(k^2 + 8k + 12) \\ &= k^3 + 8k^2 + 12k + k^2 + 8k + 12 = k^3 + 9k^2 + 20k + 12 \\ &= (3a - 6k^2 - 5k) + 9k^2 + 20k + 12 \end{aligned}$$

[By substituting for  $k^3$  from (ii)]

$$\begin{aligned} &= 3a + 3k^2 + 15k + 12 \\ &= 3(a + k^2 + 5k + 4) \\ &= 3b \text{ where } b = (a + k^2 + 5k + 4) \in \mathbb{Z} \end{aligned}$$

$\Rightarrow (k + 1)(k + 2)(k + 6)$  is a multiple of 3.

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**20.  $10^{2n-1} + 1$  is divisible by 11.**

**Sol.** Let  $P(n)$  :  $10^{2n-1} + 1$  is divisible by 11 ... (i)

**Step I:** Putting  $n = 1$  in (i),

$P(1)$  :  $10^1 + 1$  is divisible by 11 or 11 is divisible by 11, which is true. Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

Putting  $n = k$  in (i),

i.e., let  $10^{2k-1} + 1$  be divisible by 11

i.e.,  $10^{2k-1} + 1 = 11a$  where  $a \in \mathbb{Z}$ .

$$\Rightarrow 10^{2k-1} = 11a - 1 \quad \dots(ii)$$

**Step III:** We want to prove that

Putting  $n = (k + 1)$  in (i),

$P(k + 1)$  :  $10^{2(k+1)-1} + 1 = 10^{2k+1} + 1$  is divisible by 11, is also true.

Now,  
 $10^{2k+1} + 1 = 10^{(2k-1)+2} + 1$

$$\begin{aligned}
 &= 10^{2k-1} \cdot 10^2 + 1 \\
 &= 100(11a - 1) + 1 && \text{[Using (ii)]} \\
 &= 1100a - 99 = 11(100a - 9) \\
 &= 11b \text{ where } b = (100a - 9) \in \mathbb{Z}
 \end{aligned}$$

$\Rightarrow 10^{2k+1} + 1$  is divisible by 11.

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**21.  $x^{2n} - y^{2n}$  is divisible by  $x + y$ .**

**Sol.** Let  $P(n) : x^{2n} - y^{2n}$  is divisible by  $x + y$ . ... (i)

**Step I:** Putting  $n = 1$  in (i),

$P(1) : x^2 - y^2$  is divisible by  $x + y$

or  $(x + y)(x - y)$  is divisible by  $x + y$ , which is true.

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

i.e., let  $x^{2k} - y^{2k}$  be divisible by  $x + y$

i.e.,  $x^{2k} - y^{2k} = a(x + y)$  where  $a \in \mathbb{Z}$

$\Rightarrow x^{2k} = a(x + y) + y^{2k}$  ... (ii)

**Step III:** We want to prove that

$P(k+1) : x^{2k+2} - y^{2k+2}$  is divisible by  $x + y$ , is also true.

Now,  $x^{2k+2} - y^{2k+2}$

$$\begin{aligned}
 &= x^2 \cdot x^{2k} - y^2 \cdot y^{2k} \\
 &= x^2 [a(x + y) + y^{2k}] - y^2 \cdot y^{2k} \text{ [Using (ii)]} \\
 &= ax^2(x + y) + x^2 \cdot y^{2k} - y^2 \cdot y^{2k} \\
 &= ax^2(x + y) + y^{2k}(x^2 - y^2) \\
 &= ax^2(x + y) + y^{2k}(x + y)(x - y) \\
 &= (x + y)[ax^2 + y^{2k}(x - y)] \\
 &= b(x + y) \text{ where } b = [ax^2 + y^{2k}(x - y)] \in \mathbb{Z}
 \end{aligned}$$

$\Rightarrow x^{2k+2} - y^{2k+2}$  is divisible by  $x + y$

$\Rightarrow P(k+1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**22.  $3^{2n+2} - 8n - 9$  is divisible by 8.**

**Sol.** Let  $P(n) : 3^{2n+2} - 8n - 9$  is divisible by 8. ... (i)

**Step I :** Putting  $n = 1$  in (i)

$P(1) : 3^4 - 8 - 9 = 81 - 17$  is divisible by 8  
or  $64$  is divisible by 8, which is true.

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

*i.e.*, let  $3^{2k+2} - 8k - 9$  be divisible by 8

*i.e.*,  $3^{2k+2} - 8k - 9 = 8a$  where  $a \in \mathbb{Z}$

*i.e.*,  $3^{2k+2} = 8a + 8k + 9$  ... (ii)

**Step III:** We want to prove that [Putting  $n = k + 1$  in (i)]

$P(k + 1) : 3^{2(k+1)+2} - 8(k + 1) - 9$  is divisible by 8, is also true.

$$\begin{aligned} \text{Now, } 3^{2k+4} - 8(k+1) - 9 &= 3^{(2k+4)} - 8k - 8 - 9 \\ &= 3^{(2k+2)+2} - 8k - 8 - 9 \\ &= 3^{2k+2} \cdot 3^2 - 8k - 17 \\ &= 9(8a + 8k + 9) - 8k - 17 \quad [\text{Using (ii)}] \\ &= 72a + 72k + 81 - 8k - 17 \\ &= 72a + 64k + 64 = 8(9a + 8k + 8) \\ &= 8b, \text{ where } b = 9a + 8k + 8 \in \mathbb{Z} \end{aligned}$$

$\Rightarrow 3^{2k+4} - 8(k+1) - 9$  is divisible by 8.

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence, by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**23.  $41^n - 14^n$  is a multiple of 27.**

**Sol.** Let  $P(n) : 41^n - 14^n$  is a multiple of 27. ... (i)

**Step I:** Putting  $n = 1$  in (i),

$P(1) : 41 - 14$  is a multiple of 27

or  $27$  is a multiple of 27, which is true.

Thus,  $P(1)$  is true.

**Step II:** Assume that  $P(k)$  is true for some natural number  $k$ .

Putting  $n = k$  in (i)

*i.e.*, let  $41^k - 14^k$  be a multiple of 27

*i.e.*,  $41^k - 14^k = 27a$ , where  $a \in \mathbb{Z}$

$\Rightarrow 41^k = 27a + 14^k$  ... (ii)



**Step III:** We want to prove that [Putting  $n = k + 1$  in (i)]

$P(k + 1) : 41^{k+1} - 14^{k+1}$  is a multiple of 27 is also true.

$$\begin{aligned} \text{Now, } 41^{k+1} - 14^{k+1} &= 41 \cdot 41^k - 14 \cdot 14^k \\ &= 41(27a + 14^k) - 14 \cdot 14^k \quad [\text{Using (ii)}] \\ &= 41 \times 27a + 41 \cdot 14^k - 14 \cdot 14^k \\ &= 41 \times 27a + 27 \cdot 14^k = 27(41a + 14^k) \\ &= 27b, \text{ where } b = (41a + 14^k) \in \mathbb{Z} \end{aligned}$$

$\Rightarrow 41^{k+1} - 14^{k+1}$  is a multiple of 27.

$\Rightarrow P(k + 1)$  is true when  $P(k)$  is true. Hence by P.M.I.,  $P(n)$  is true for every positive integer  $n$ .

**24.  $2n + 7 < (n + 3)^2$ .**

**Sol.** Let  $P(n)$  be the statement " $2n + 7 < (n + 3)^2 \dots$  (i)

**Step I:** Then  $P(1)$  is the statement " $2 \times 1 + 7 < (1 + 3)^2$  or  $9 < 16$ " which is true. Thus  $P(1)$  is true.

**Step II:** Suppose  $P(k)$  is true, [Putting  $n = k$  in (i)],

$$\text{then } 2k + 7 < (k + 3)^2 \quad \dots (ii)$$

**Step III:**  $P(k + 1)$  is the statement " $2(k + 1) + 7 < (k + 4)^2$   
 $= k^2 + 8k + 16$ "

$$\begin{aligned} \text{Now, } 2(k + 1) + 7 &= (2k + 7) + 2 \\ &< (k + 3)^2 + 2 \quad [\text{Using (ii)}] \\ &= k^2 + 9 + 6k + 2 = k^2 + 6k + 11 \end{aligned}$$

Adding and subtracting  $(2k + 5)$  in the R.H.S.;

$$\begin{aligned} &= k^2 + 6k + 11 + 2k + 5 - (2k + 5) \\ &= (k^2 + 8k + 16) - (2k + 5) \\ &= (k + 4)^2 - (2k + 5) \\ &< (k + 4)^2, \text{ since } 2k + 5 > 0 \text{ for all } k \in \mathbb{N} \end{aligned}$$

$\Rightarrow P(k + 1)$  is true.

$\therefore$  By the principle of mathematical induction,  $P(n)$  is true for all  $n \in \mathbb{N}$ .

