8 Binomial Theorem

Lesson at a Glance

1. An algebraic expression of two terms which are connected by the positive (+) or negative (–) sign is called a **Binomial expression.**

   For example, $2x + 3y$, $2x - \frac{1}{3x}$ are binomial expressions.

2. **Binomial theorem for the expansion of $(x + y)^n$, $(n \in N)$**

   $$(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \ldots {}^nC_{n-1}\, x^1 y^{n-1} + {}^nC_n\, y^n.$$

3. General term T_{r+1} *of binomial expansion* $(x + y)^n$; $n \in N$ is

   $$T_{r+1} = {}^nC_r\, x^{n-r} y^r.$$

4. Number of terms in the binomial expansion $(x + y)^n$; $(n \in N)$ is $n + 1$.

5. $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n$. $(n \in N)$

6. **T_{r+1} of $(1 + x)^n$ is ${}^nC_r\, x^r$.**

7. Coefficient of x^r in $(1 + x)^n$ *is* ${}^nC_r$

8. $(x - y)^n$ $(n \in N)$

   $$= {}^nC_0\, x^n - {}^nC_1\, x^{n-1} y + {}^nC_2\, x^{n-2} y^2 - \ldots + (-1)^n\, {}^nC_n\, y^n.$$

9. **pth term from the end in the expansion of $(x + y)^n$; $n \in N$**

   Method I. pth term from the end in the expansion of $(x + y)^n$ is pth term from the beginning in $(y + x)^n$.

   Method II. pth term from the end in the expansion of $(x + y)^n$ *is* $(n - p + 2)$th from the beginning *i.e.*, T_{n-p+2}.

10. **Middle term(s) of binomial expansion $(x + y)^n$, $(n \in N)$.**

Case I. n is even

There is only one middle term $T_{\frac{n}{2}+1}$.

Case II. n is odd

Then there are two middle terms $T_{\frac{n+1}{2}}$ and next term.

11. Greatest binomial coefficient

Coefficient of middle term(s) in the expansion of $(x + y)^n$; $(n \in N)$ is greatest binomial coefficient.

Therefore,

Case I. If n is even, then the greatest binomial coefficient is ${}^nC_{\frac{n}{2}}$.

Case II. If n is odd, then the greatest binomial coefficient

is ${}^nC_{\frac{n-1}{2}}$ *or* ${}^nC_{\frac{n+1}{2}}\left(= {}^nC_{\frac{n-1}{2}}\right)$. $\left(\because {}^nC_r = {}^nC_{n-r}\right)$

12. To find the term independent of x or absolute term or constant term

⇒ To find the term containing x^0.

TEXTBOOK QUESTIONS SOLVED

EXERCISE 8.1 (*Page No.: 166–167*)

Expand each of the expressions in Exercises 1 to 5.

1. $(1 - 2x)^5$

Sol. $(1 - 2x)^5 = {}^5C_0\,(1)^5 - {}^5C_1\,(1)^4\,(2x) + {}^5C_2\,(1)^3\,(2x)^2 - {}^5C_3\,(1)^2\,(2x)^3 + {}^5C_4\,(1)\,(2x)^4 - {}^5C_5\,(2x)^5$

[∵ The terms in the Binomial expansion $(x-y)^n$ are alternately positive and negative]

$= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - 32x^5$

$$\left[\because\ {}^5C_5 = {}^5C_0 = 1,\ {}^5C_4 = {}^5C_1 = 5,\ {}^5C_3 = {}^5C_2 = \frac{5\times 4}{2\times 1} = 10\right]$$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5.$

2. $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Sol. $\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$

$- {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$

$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right)$

$- 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$

$\left[\because\ {}^5C_5 = {}^5C_0 = 1,\ {}^5C_4 = {}^5C_1 = 5,\ {}^5C_3 = {}^5C_2 = \frac{5\times4}{2\times1} = 10\right]$

$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{1}{32}x^5.$

3. $(2x-3)^6$

Sol. $(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2$
$- {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6$

$= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27)$
$+ 15(4x^2)(81) - 6(2x)(243) + 729$

$\left[\because\ {}^6C_6 = {}^6C_0 = 1,\ {}^6C_5 = {}^6C_1 = 6,\ {}^6C_4 = {}^6C_2 = \frac{6\times5}{2\times1} = 15,\right.$

$\left. {}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20\right]$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729.$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Sol. $\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2$

$+ {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$

$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right)$$

$$+ 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$

$$\left[\because \ {}^5C_5 = {}^5C_0 = 1, {}^5C_4 = {}^5C_1 = 5, {}^5C_3 = {}^5C_2 = \frac{5\times4}{2\times1} = 10\right]$$

$$= \frac{1}{243}x^3 + \frac{5}{81}x^3 + \frac{10}{27}x + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}.$$

5. $\left(x+\frac{1}{x}\right)^6$

Sol. $\left(x+\frac{1}{x}\right)^6 = {}^6C_0\,x^6 + {}^6C_1\,x^5\left(\frac{1}{x}\right) + {}^6C_2\,x^4\left(\frac{1}{x}\right)^2 + {}^6C_3\,x^3\left(\frac{1}{x}\right)^3$

$$+ {}^6C_4\,x^2\left(\frac{1}{x}\right)^4 + {}^6C_5\,x\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}.$$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$

Sol. $(96)^3 = (100 - 4)^3$

$= {}^3C_0\,(100)^3 - {}^3C_1\,(100)^2\,(4) + {}^3C_2\,(100)\,(4)^2 - {}^3C_3\,(4)^3$

$= 1000000 - 3(40000) + 3(1600) - 64$

$[\because \ {}^3C_3 = {}^3C_0 = 1, {}^3C_2 = {}^3C_1 = 3]$

$= 1000000 - 120000 + 4800 - 64$

$= 1004800 - 120064 = 884736.$

7. $(102)^5$

Sol. $(102)^5 = (100 + 2)^5$

$= {}^5C_0\,(100)^5 + {}^5C_1\,(100)^4\,(2) + {}^5C_2\,(100)^3\,(2)^2$

$+ {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 + {}^5C_5\,(2)^5$

$= 10000000000 + 5(200000000) + 10(4000000)$

$+ 10(80000) + 5(1600) + 32$

$$\left[\because \ {}^5C_5 = {}^5C_0 = 1,\ {}^5C_4 = {}^5C_1 = 5,\ {}^5C_3 = {}^5C_2 = \frac{5\times 4}{2\times 1} = 10\right]$$

$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$

$= 11040808032.$

8. $(101)^4$

Sol. $(101)^4 = (100 + 1)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1\,(100)^3 + {}^4C_2\,(100)^2 + {}^4C_3\,(100) + {}^4C_4$

$= 100000000 + 4(1000000) + 6(10000) + 4(100) + 1$

$$\left[\because \ {}^4C_4 = {}^4C_0 = 1,\ {}^4C_3 = {}^4C_1 = 4,\ {}^4C_2 = \frac{4\times 3}{2\times 1} = 6\right]$$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401.$

9. $(99)^5$

Sol. $(99)^5 = (100 - 1)^5$

$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4 + {}^5C_2\,(100)^3 - {}^5C_3\,(100)^2 + {}^5C_4\,(100) - {}^5C_5$

$= 10000000000 - 5(100000000) + 10(1000000) - 10(10000) + 5(100) - 1$

$$\left[\because \ {}^5C_5 = {}^5C_0 = 1,\ {}^5C_4 = {}^5C_1 = 5,\ {}^5C_3 = {}^5C_2 = \frac{5\times 4}{2\times 1} = 10\right]$$

$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$

$= 10010000500 - 500100001$

$= 9509900499.$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000?

Sol. $(1.1)^{10000} = (1 + 0.1)^{10000}$

$= {}^{10000}C_0 + {}^{10000}C_1\,(0.1)$ + other positive terms

$= 1 + 10000 \times 0.1$ + other positive terms

$= 1 + 1000$ + other positive terms

> 1000

$\Rightarrow\ (1.1)^{10000} > 1000.$

11. Find $(a + b)^4 - (a - b)^4$. Hence, evaluate

$$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4.$$

Sol. Expanding by binomial theorem

$$(a+b)^4-(a-b)^4 = ({}^4C_0\, a^4 + {}^4C_1\, a^3b + {}^4C_2\, a^2b^2 + {}^4C_3\, ab^3 + {}^4C_4\, b^4) - ({}^4C_0\, a^4 - {}^4C_1\, a^3b + {}^4C_2\, a^2b^2 - {}^4C_3\, ab^3 + {}^4C_4\, b^4)$$

$$= 2\cdot {}^4C_1\, a^3b + 2\cdot {}^4C_3\, ab^3$$

$$= 2\cdot 4\, a^3b + 2\cdot 4\cdot ab^3 \qquad [\because {}^4C_3 = {}^4C_1 = 4]$$

or $$(a+b)^4-(a-b)^4 = 8ab\,(a^2+b^2) \qquad ...(i)$$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$ on both sides of (i),

$$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8\sqrt{3}\ \sqrt{2}\,(3+2)$$

$$= 40\sqrt{6}.$$

12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.

Sol. $(x + 1)^6 + (x - 1)^6$

$$= [{}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6] + [{}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6]$$

$$= 2({}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6)$$

$$= 2(x^6 + 15x^4 + 15x^2 + 1)$$

$$\left[\because\ {}^6C_6 = {}^6C_0 = 1,\ {}^6C_4 = {}^6C_2 = \frac{6\times 5}{2\times 1} = 15\right]$$

$\therefore\ (x + 1)^6 + (x - 1)^6 = 2(x^6 + 15x^4 + 15x^2 + 1)$

Putting $x = \sqrt{2}$, we get

$$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = 2[(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1]$$

$$= 2[8 + 15(4) + 15(2) + 1]$$

$$= 2(99) = 198.$$

13. Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Sol. $$9^{n+1} = (1 + 8)^{n+1}$$

$$= {}^{n+1}C_0 + {}^{n+1}C_1\cdot 8 + {}^{n+1}C_2\cdot 8^2 + {}^{n+1}C_3\cdot 8^3 + \ldots + {}^{n+1}C_{n+1}\cdot 8^{n+1}$$

$[\because (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 ... + {}^nC_n x^n]$

or $$9^{n+1} = 1 + (n+1).8 + {}^{n+1}C_2 .8^2 + {}^{n+1}C_3 . 8^3 + ... + {}^{n+1}C_{n+1}.8^{n+1}$$

Transposing the first two terms of R.H.S. to L.H.S.; we have

$$9^{n+1} - 8n - 9 = 8^2[{}^{n+1}C_2 + 8 . {}^{n+1}C_3 + ... + 8^{n-1}]$$

$$[\because 8^{n+1} = 8^{n-1+1+1} = 8^{n-1+2} = 8^{n-1}.8^2]$$

$$= 64 \times \text{a positive integer}$$

$\therefore 9^{n+1} - 8n - 9$ is divisible by 64.

14. Prove that $\sum_{r=0}^{n} 3^r \; {}^nC_r = 4^n$.

Sol. We know that for $n \in N$,

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n \quad ...(i)$$

Putting $x = 3$ on both sides of eqn. (i), we have

$$(1+3)^n = 4^n = {}^nC_0 + {}^nC_1 \cdot 3 + {}^nC_2 \cdot 3^2 + ... + {}^nC_r 3^r + ... + {}^nC_n \cdot 3^n$$

or $$4^n = \sum_{r=0}^{n} 3^r . {}^nC_r$$

or $$\sum_{r=0}^{n} {}^nC_r . 3^r = 4^n .$$

EXERCISE 8.2 (*Page No.: 171*)

Find the coefficient of

1. x^5 in $(x + 3)^8$

Sol. Suppose x^5 occurs in the $(r + 1)$th term of the expansion of $(x + 3)^8$.

Now, $T_{r+1} = {}^8C_r . x^{8-r} . 3^r, 0 \le r \le 8.$ ... (i)

$[\because T_{r+1}$ of $(x+y)^n = {}^nC_r x^{n-r} y^r]$

It will contain x^5 if $8 - r = 5$, *i.e.*, if $r = 3$

Putting $r = 3$ in (i). $T_4 = {}^8C_3 x^5 \cdot 3^3$

$$\therefore \text{ Coefficient of } x^5 \text{ is } {}^8C_3 . 3^3 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times 27$$

$$= 56 \times 27 = 1512.$$

2. a^5b^7 in $(a - 2b)^{12}$.

Sol. Suppose a^5b^7 occurs in the $(r + 1)$th term of the expansion of $(a - 2b)^{12}$.

Now, $T_{r+1} = {}^{12}C_r \,.\, a^{12-r} \,.\, (-2b)^r$

$= {}^{12}C_r \,.\, (-2)^r \,.\, a^{12-r} \,.\, b^r$ ... (*i*)

It will involve a^5b^7 if $12 - r = 5$ and $r = 7$, *i.e.*, if $r = 7$

Putting $r = 7$ in (*i*), $T_8 = {}^{12}C_7\,(-2)^7 a^5b^7$

$\therefore$ Coefficient of a^5b^7 is ${}^{12}C_7\,(-2)^7 = {}^{12}C_5\,(-2^7)$

$$= -\frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} \times 2^7$$

$= -792 \times 128$

$= -101376.$

Write the general term in the expansion of

3. $(x^2 - y)^6$

Sol. General term is $T_{r+1} = {}^6C_r\,(x^2)^{6-r}\,(-y)^r$

[$\because$ General Term T_{r+1} of $(x + y)^n$ is ${}^nC_r\, x^{n-r}y^r$]

$= {}^6C_r \,.\, x^{12-2r} \,.\, (-1)^r\, y^r$

$= (-1)^r \,.\, {}^6C_r \,.\, x^{12-2r} \,.\, y^r.$

4. $(x^2 - yx)^{12}$, $x \neq 0$

Sol. General term is $T_{r+1} = {}^{12}C_r \,.\, (x^2)^{12-r} \,.\, (-yx)^r$

$= {}^{12}C_r \,.\, x^{24-2r} \,.\, (-1)^r \,.\, y^r \,.\, x^r$

$= (-1)^r \,.\, {}^{12}C_r \,.\, x^{24-r} \,.\, y^r.$ ($\because 24 - 2r + r = 24 - r$)

5. Find the 4th term in the expansion of $(x - 2y)^{12}$.

Sol. $T_4 = T_{3+1}$ (Here $r = 3$, $n = 12$)

$= {}^{12}C_3\, x^{12-3} \,.\, (-2y)^3$

$$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} \,.\, x^9 \,.\, (-8y^3)$$

$= -220 \times 8x^9y^3 = -1760x^9y^3.$

6. Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$.

Sol. $T_{13} = T_{12+1}$ (Here $r = 12$, $n = 18$)

$$= {}^{18}C_{12}\,(9x)^{18-12}\,.\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= {}^{18}C_6\,.\,(9x)^6\,.\left(\frac{1}{3}\right)^{12}.\left(\frac{1}{\sqrt{x}}\right)^{12}$$

$[\because\ {}^nC_r = {}^nC_{n-r}, (-1)^{12} = +1]$

$$= \frac{18\times17\times16\times15\times14\times13}{6\times5\times4\times3\times2\times1}\,.\,(3^2)^6\,.\,x^6\,.\,\frac{1}{3^{12}}\,.\,\frac{1}{x^6}$$

$$= 18564 \times 3^{12} \times \frac{1}{3^{12}} = 18564.$$

Find the middle terms in the expansions of:

7. $\left(3-\frac{x^3}{6}\right)^7$

Sol. In the expansion of $\left(3-\frac{x^3}{6}\right)^7$, $n = 7$ is odd. Therefore, there are two middle terms: $T_{\frac{n+1}{2}} = T_{\frac{7+1}{2}} = T_4$ and next term T_5

$$T_4 = {}^7C_3\,(3)^4\left(-\frac{x^3}{6}\right)^3 = \frac{7\times6\times5}{3\times2\times1}\times 81\times\left(-\frac{x^9}{216}\right)$$

$$= -35\times\frac{3}{8}x^9 = -\frac{105}{8}x^9$$

and $$T_5 = {}^7C_4\,(3)^3\left(-\frac{x^3}{6}\right)^4 = {}^7C_3\,(27)\left(\frac{x^{12}}{1296}\right)$$

$$= \frac{7\times6\times5}{3\times2\times1}\times\frac{x^{12}}{48} = \frac{35}{48}x^{12}.$$

8. $\left(\frac{x}{3}+9y\right)^{10}$.

Sol. In the expansion of $\left(\frac{x}{3}+9y\right)^{10}$, $n = 10$ is even. Therefore there is only one middle term, namely $T_{\frac{n}{2}+1} = T_{\frac{10}{2}+1} = T_6$

$$T_6 = {}^{10}C_5 \left(\frac{x}{3}\right)^5 . (9y)^5 = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} . \frac{x^5}{3^5} (3^2)^5 y^5$$

$$= 252 \times \frac{3^{10}}{3^5} x^5y^5 = 252 \times 3^5x^5y^5$$

$$= 252 \times 243x^5y^5 = 61236x^5y^5.$$

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Sol. In the expansion of $(1 + a)^{m+n}$, the general term is

$T_{r+1} = {}^{m+n}C_r\ a^r$

$[\because T_{r+1}$ of $(1+x)^n$ is ${}^nC_r x^r]$

$\therefore$ Coefficient of a^r is ${}^{m+n}C_r$. ...(*i*)

Putting r = m and r = n in (*i*),

$$\text{Coefficient of } a^m = {}^{m+n}C_m = \frac{(m+n)!}{m!n!} \quad ...(ii)$$

$$\text{and coefficient of } a^n = {}^{m+n}C_n = \frac{(m+n)!}{n!m!} \quad ...(iii)$$

From (*ii*) and (*iii*), we have

Coefficient of a^m = Coefficient of a^n.

10. The coefficients of the $(r - 1)$th, rth and $(r + 1)$th terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

Sol. In the expansion of $(x + 1)^n$, the general term is

$$T_{r+1} = {}^nC_r\, x^{n-r}. (1)^r = {}^nC_r\, x^{n-r}$$

$\Rightarrow$ Coefficient of $(r + 1)$ th term is ${}^nC_r$.

Changing r to $r - 1$ and $r - 2$, coefficient of rth term is ${}^nC_{r-1}$ and coefficient of $(r - 1)$th term is ${}^nC_{r-2}$.

Since coefficients of $(r - 1)$th, rth and $(r + 1)$th terms are in the ratio 1 : 3 : 5.

$\therefore\ {}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1 : 3 : 5.$

$\Rightarrow\ {}^nC_{r-2} : {}^nC_{r-1} = 1 : 3$ ...(*i*)

and ${}^nC_{r-1} : {}^nC_r = 3 : 5$ ...(*ii*)

From (*i*), $\frac{n!}{(r-2)!(n-r+2)!} : \frac{n!}{(r-1)!(n-r+1)!} = 1 : 3$

$\Rightarrow \quad \frac{(r-1)!}{(r-2)!} \cdot \frac{(n-r+1)!}{(n-r+2)!} = \frac{1}{3}$

$\Rightarrow \quad \frac{(r-1)(r-2)!}{(r-2)!} \cdot \frac{(n-r+1)!}{(n-r+2)(n-r+1)!} = \frac{1}{3}$

$[\because r-1 > r-2 \text{ and } n-r+2 > n-r+1]$

$\Rightarrow \quad \frac{r-1}{n-r+2} = \frac{1}{3}$

$\Rightarrow \quad 3r - 3 = n - r + 2$

$\Rightarrow n - 4r + 5 = 0 \quad ...(iii)$

From (*ii*), $\frac{n!}{(r-1)!(n-r+1)!} : \frac{n!}{r!(n-r)!} = \frac{3}{5}$

$\Rightarrow \quad \frac{r!}{(r-1)!} \cdot \frac{(n-r)!}{(n-r+1)!} = \frac{3}{5}$

$\Rightarrow \quad \frac{r(r-1)!}{(r-1)!} \cdot \frac{(n-r)!}{(n-r+1)(n-r)!} = \frac{3}{5}$

$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{5}$

$\Rightarrow \quad 5r = 3n - 3r + 3$

$\Rightarrow \quad 3n - 8r + 3 = 0 \quad ...(iv)$

Multiplying (*iii*) by 2, we have

$2n - 8r + 10 = 0 \quad ...(v)$

Subtracting (*v*) from (*iv*), $n - 7 = 0 \quad \therefore \quad n = 7$

Putting $n = 7$ in (*iii*), $7 - 4r + 5 = 0 \Rightarrow -4r = -12 \therefore r = 3$

Hence $n = 7$, $r = 3$.

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Sol. In the expansion of $(1 + x)^{2n}$, the general term is $T_{r+1} = {}^{2n}C_r\, x^r$.

$\Rightarrow$ Coefficient of x^r is ${}^{2n}C_r$

Changing r to n, coefficient of x^n is

$$^{2n}C_n = \frac{2n!}{n!\,n!} = \frac{2n\,(2n-1)!}{n!\,.\,n(n-1)!}$$

$$= 2\left(\frac{(2n-1)!}{n!\,.\,(n-1)!}\right) \quad ...(i)$$

In the expansion of $(1 + x)^{2n-1}$, the general term is

$T_{r+1} = {}^{2n-1}C_r\, x^r$

$\Rightarrow$ Coefficient of x^r is ${}^{2n-1}C_r$

Changing r to n, coefficient of x^n is

$$^{2n-1}C_n = \frac{(2n-1)!}{n!\,(n-1)!} \quad ...(ii)$$

From (i) and (ii), it follows that the coefficient of x^n in the expansion of $(1 + x)^{2n}$.

= 2 × coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Sol. $(1 + x)^m = {}^mC_0 + {}^mC_1 x + {}^mC_2\, x^2 + ... + {}^mC_m\, x^m$.

$\therefore$ Coefficient of $x^2 = {}^mC_2$

Given: coefficient of $x^2 = 6$

$\therefore \quad {}^mC_2 = 6 \quad \Rightarrow \quad \frac{m(m-1)}{2\times 1} = 6$

$\Rightarrow \quad m^2 - m - 12 = 0 \quad \Rightarrow \quad (m - 4)(m + 3) = 0$

$\Rightarrow \quad m = 4, -3$

$\therefore$ The required positive value of m is 4.

MISCELLANEOUS EXERCISE ON CHAPTER 8

(Page No.: 175–176)

1. Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Sol. In the expansion of $(a + b)^n$, we are given that

$T_1 = 729 \quad \Rightarrow \quad {}^nC_0 a^n = 729$

$\Rightarrow \quad a^n = 729 \quad ...(i)$

$T_2 = 7290 \quad \Rightarrow \quad {}^nC_1 a^{n-1} b = 7290$

$\Rightarrow \quad na^{n-1}b = 7290 \quad ...(ii)$

$T_3 = 30375 \Rightarrow {}^nC_2 a^{n-2}b^2 = 30375$

$\Rightarrow \dfrac{n(n-1)}{2\times 1} a^{n-2}b^2 = 30375$

$\Rightarrow n(n-1)a^{n-2}b^2 = 60750$ ...(*iii*)

Dividing (*ii*) by (*i*), we get

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729} \Rightarrow \frac{b}{a} = \frac{10}{n} \quad ...(iv)$$

$[\because a^n = a^{n-1+1} = a^{n-1}a]$

Dividing (*iii*) by (*ii*), we get

$$\frac{n(n-1)a^{n-2}b^2}{na^{n-1}b} = \frac{60750}{7290} \Rightarrow \frac{b}{a} = \frac{25}{3(n-1)} \quad ...(v)$$

From (*iv*) and (*v*), equating the two values of $\dfrac{b}{a}$, we have

$$\frac{10}{n} = \frac{25}{3(n-1)}$$

$\Rightarrow \dfrac{2}{n} = \dfrac{5}{3(n-1)} \Rightarrow 6(n-1) = 5n$

$\Rightarrow 6n - 6 = 5n \qquad \therefore n = 6$

Putting $n = 6$ in (*i*), $a^6 = 729 = 3^6$

$\Rightarrow a = 3$

Putting $n = 6$ and $a = 3$ in (*ii*),

$6 \times 3^5 b = 3^6 \times 10 = 3^5 \times 3 \times 2 \times 5$

$\Rightarrow b = 5$

Hence $a = 3, b = 5, n = 6.$

2. Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Sol. In the expansion of $(3 + ax)^9$, the general term is

$T_{r+1} = {}^9C_r . 3^{9-r} . (ax)^r = {}^9C_r . 3^{9-r} a^r x^r$

$\Rightarrow$ Coefficient of x^r is ${}^9C_r . 3^{9-r} . a^r$

Putting $r = 2$ and 3, we have

coefficient of $x^2 = {}^9C_2 . 3^7 a^2$

and coefficient of $x^3 = {}^9C_3 . 3^6 a^3$

Since coefficients of x^2 and x^3 are given to be equal.

$\therefore \quad {}^9C_2 . 3^7 a^2 = {}^9C_3 . 3^6 a^3$

Dividing both sides by $3^6 a^2$

$$\Rightarrow \quad \frac{9\times8}{2\times1}\times 3 = \frac{9\times8\times7}{3\times2\times1}\times a$$

$$\Rightarrow \quad 108 = 84\, a \Rightarrow a = \frac{108}{84} = \frac{9}{7}$$

3. Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Sol. $(1 + 2x)^6 (1 - x)^7$

$$= [{}^6C_0 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6]\,[{}^7C_0 - {}^7C_1 x + {}^7C_2 x^2 - {}^7C_3 x^3 + {}^7C_4 x^4 - {}^7C_5 x^5 + {}^7C_6 x^6 - {}^7C_7 x^7]$$

$$= [1 + 6(2x) + 15(4x^2) + 20(8x^3) + 15(16x^4) + 6(32x^5) + 64x^6]\,[1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7]$$

$$\left[\because\ {}^6C_6 = {}^6C_0 = 1,\ {}^6C_5 = {}^6C_1 = 6,\ {}^6C_4 = {}^6C_2 = \frac{6\times5}{2\times1} = 15,\right.$$

$${}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20,\ {}^7C_7 = {}^7C_0 = 1,\ {}^7C_6 = {}^7C_1 = 7,$$

$$\left.{}^7C_5 = {}^7C_2 = \frac{7\times6}{2\times1} = 21,\ {}^7C_4 = {}^7C_3 = \frac{7\times6\times5}{3\times2\times1} = 35\right]$$

$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6) \times (1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$$

The terms containing x^5 in this product are

$$= (1)(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1)$$

$$= (-21 + 420 - 2100 + 3360 - 1680 + 192)x^5$$

$$= 171x^5$$

$\therefore$ Coefficient of x^5 is 171.

4. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

Sol. We know that $a = a - b + b$

$$\therefore \quad a^n = [(a-b)+b]^n$$

Expanding R.H.S. of the form $(x + y)^n$ by Binomial Theorem,

$$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1} b + \ldots + {}^nC_{n-1}(a-b)\, b^{n-1} + {}^nC_n\, b^n$$

Bringing the last term b^n [$\because$ ${}^nC_n = 1$] to the L.H.S., we have

$$a^n - b^n = (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)\,b^{n-1}$$

$$= (a-b)[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}]$$

$$= (a-b)\,(\text{an integer})$$

$\therefore$ $a^n - b^n$ is divisible by $a - b$.

5. Evaluate $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$.

Sol. Putting $\sqrt{3} = a$ and $\sqrt{2} = b$, the given expression is

$$= (\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$$

$$= (a + b)^6 - (a - b)^6$$

$$= [{}^6C_0\,a^6 + {}^6C_1\,a^5b + {}^6C_2\,a^4b^2 + {}^6C_3\,a^3b^3 + {}^6C_4\,a^2b^4 + {}^6C_5\,ab^5 + {}^6C_6\,b^6] - [{}^6C_0\,a^6 - {}^6C_1\,a^5b + {}^6C_2\,a^4b^2 - {}^6C_3\,a^3b^3 + {}^6C_4\,a^2b^4 - {}^6C_5\,ab^5 + {}^6C_6\,b^6]$$

$$= 2({}^6C_1\,a^5b + {}^6C_3\,a^3b^3 + {}^6C_5\,ab^5)$$

$$= 2(6a^5b + 20a^3b^3 + 6ab^5)$$

$$\left[\because\ {}^6C_5 = {}^6C_1 = 6,\ {}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20\right]$$

$$= 4ab(3a^4 + 10a^2b^2 + 3b^4)$$

$\therefore$ $(a + b)^6 - (a - b)^6 = 4ab(3a^4 + 10a^2b^2 + 3b^4)$ ...(*i*)

Putting back $a = \sqrt{3}$ and $b = \sqrt{2}$, in (*i*), we get

$$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$$

$$= 4\sqrt{3}\,\sqrt{2}\,[3(\sqrt{3})^4 + 10(\sqrt{3})^2(\sqrt{2})^2 + 3(\sqrt{2})^4]$$

$$= 4\sqrt{6}\,[3(9) + 10(3)(2) + 3(4)] \quad \left[\because (\sqrt{t})^4 = (t^{1/2})^4 = t^2\right]$$

$$= 4\sqrt{6}\,(27 + 60 + 12)$$

$$= 4\sqrt{6}\,(99) = 396\sqrt{6}.$$

6. Find the value of $(a^2 + \sqrt{a^2-1})^4 + (a^2 - \sqrt{a^2-1})^4$.

Sol. Putting $a^2 = x$ and $\sqrt{a^2-1} = y$, the given expression is

$= (x + y)^4 + (x - y)^4$

$= [{}^4C_0\, x^4 + {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 + {}^4C_3\, xy^3 + {}^4C_4\, y^4]$

$\quad + [{}^4C_0\, x^4 - {}^4C_1\, x^3y + {}^4C_2\, x^2y^2 - {}^4C_3\, xy^3 + {}^4C_4\, y^4]$

$= 2\,[{}^4C_0\, x^4 + {}^4C_2\, x^2y^2 + {}^4C_4\, y^4]$

$= 2(x^4 + 6x^2y^2 + y^4)$

$$\left[\because\ {}^4C_4 = {}^4C_0 = 1,\ {}^4C_2 = \frac{4\times 3}{2\times 1} = 6\right]$$

$\therefore\quad (x + y)^4 + (x - y)^4 = 2(x^4 + 6x^2y^2 + y^4)$ ...(*i*)

Putting back $x = a^2$ and $y = \sqrt{a^2-1}$, in (*i*), we have

$(a^2 + \sqrt{a^2-1}\,)^4 + (a^2 - \sqrt{a^2-1}\,)^4$

$= 2[(a^2)^4 + 6(a^2)^2\,(\sqrt{a^2-1}\,)^2 + (\sqrt{a^2-1}\,)^4]$

$= 2[a^8 + 6a^4\,(a^2 - 1) + (a^2 - 1)^2]$

$= 2(a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1)$

$= 2(a^8 + 6a^6 - 5a^4 - 2a^2 + 1)$

$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2.$

7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Sol. $(0.99)^5 = (1 - 0.01)^5$

Expanding by Binomial Theorem only up to first three terms,

$\approx {}^5C_0 - {}^5C_1\,(0.01) + {}^5C_2\,(0.01)^2$

$= 1 - 5(0.01) + 10(0.0001)$

$$\left[\because\ {}^5C_0 = 1,\ {}^5C_1 = 5,\ {}^5C_2 = \frac{5\times 4}{2\times 1} = 10\right]$$

$= 1 - 0.05 + 0.001$

$= 1.001 - 0.050 = 0.951$

$\therefore\quad (0.99)^5$ is approximately equal to 0.951.

8. Find *n*, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of

$\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}$: 1.

Sol. The given Binomial Expression is

$$\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n = \left(2^{1/4}+\frac{1}{3^{1/4}}\right)^n$$

put $2^{1/4} = x$ and $\frac{1}{3^{1/4}} = y$

$\therefore$ The given binomial expansion is $(x + y)^n$

According to given $\frac{\text{5th Term from beginning in } (x+y)^n}{\text{5th Term from the end in } (x+y)^n} = \frac{\sqrt{6}}{1}$

$$\Rightarrow \quad \frac{T_5 \text{ of } (x+y)^n}{T_5 \text{ of } (y+x)^n} = \sqrt{6}$$

[$\because$ p th term from end in $(x+y)^n$ = p th term from beginning in $(y+x)^n$]

$$\Rightarrow \quad \frac{{}^nC_4\, x^{n-4} y^4}{{}^nC_4\, y^{n-4} x^4} = \sqrt{6}$$

$$\Rightarrow \quad \frac{x^{n-4-4}}{y^{n-4-4}} = \sqrt{6} \Rightarrow \frac{x^{n-8}}{y^{n-8}} = \sqrt{6}$$

$$\Rightarrow \quad \left(\frac{x}{y}\right)^{n-8} = \sqrt{6}$$

Putting $x = 2^{1/4}$ and $y = \frac{1}{3^{1/4}}$, we have

$$(2^{1/4}.3^{1/4})^{n-8} = \sqrt{6} \Rightarrow (6^{1/4})^{n-8} = 6^{1/2} \qquad [\because a^k b^k = (ab)^k$$

$$\Rightarrow \quad 6^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow \frac{n-8}{4} = \frac{1}{2}$$

Cross-multiplying $2n - 16 = 4 \Rightarrow 2n = 20 \Rightarrow n = 10$

9. Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$**,** $x \neq 0$**.**

Sol. $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4 = \left[1+\left(\frac{x}{2}-\frac{2}{x}\right)\right]^4$

$$= {}^4C_0 + {}^4C_1\left(\frac{x}{2}-\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}-\frac{2}{x}\right)^2$$

$$+ {}^4C_3\left(\frac{x}{2}-\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2}-\frac{2}{x}\right)^4$$

$[\because (1+y)^n = {}^nC_0 + {}^nC_1y + {}^nC_2y^2 + {}^nC_3y^3 + \ldots\ldots\ldots + {}^nC_ny^n$

Here $y = \frac{x}{2}-\frac{2}{x}$]

$$= 1 + 4\left(\frac{x}{2}-\frac{2}{x}\right) + 6\left[\left(\frac{x}{2}\right)^2 - 2\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)+\left(\frac{2}{x}\right)^2\right]$$

$$+ 4\left[{}^3C_0\left(\frac{x}{2}\right)^3 - {}^3C_1\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right) + {}^3C_2\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^2 - {}^3C_3\left(\frac{2}{x}\right)^3\right]$$

$$+ \left[{}^4C_0\left(\frac{x}{2}\right)^4 - {}^4C_1\left(\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(\frac{x}{2}\right)\left(\frac{2}{x}\right)^3\right.$$

$$\left. {}^4C_4\left(\frac{2}{x}\right)^4\right]$$

$$= 1 + 2x - \frac{8}{x} + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right)$$

$$+ 4\left[\frac{x^3}{8} - 3\left(\frac{x^2}{4}\right)\left(\frac{2}{x}\right) + 3\left(\frac{x}{2}\right)\left(\frac{4}{x^2}\right) - \frac{8}{x^3}\right]$$

$$+ \left[\frac{x^4}{16} - 4\left(\frac{x^3}{8}\right)\left(\frac{2}{x}\right) + 6\left(\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}\right]$$

$$= 1 + 2x - \frac{8}{x} + \frac{3}{2}x^2 - 12 + \frac{24}{x^2} + \frac{x^3}{2} - 6x + \frac{24}{x} - \frac{32}{x^3}$$

$$+ \frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}$$

$$= \frac{16}{x^4} - \frac{32}{x^3} + \frac{8}{x^2} + \frac{16}{x} - 5 - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}.$$

10. Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Sol. $(3x^2 - 2ax + 3a^2)^3 = [(3x^2 - 2ax) + 3a^2]^3$

$= {}^3C_0\,(3x^2 - 2ax)^3 + {}^3C_1\,(3x^2 - 2ax)^2\,(3a^2)^1$
$+ {}^3C_2\,(3x^2 - 2ax)^1\,(3a^2)^2 + {}^3C_3\,(3a^2)^3$

$= [{}^3C_0\,(3x^2)^3 - {}^3C_1\,(3x^2)^2(2ax) + {}^3C_2\,(3x^2)(2ax)^2 - {}^3C_3\,(2ax)^3]$
$+ 3(9x^4 - 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 - 2ax)(9a^4) + 27a^6$

$= 27x^6 - 3(9x^4)(2ax) + 3(3x^2)(4a^2x^2) - 8a^3x^3$
$+ 9a^2(9x^4 - 12ax^3 + 4a^2x^2) + 27a^4(3x^2 - 2ax) + 27a^6$

$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4$
$- 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6.$