9 Sequences and Series

Lesson at a Glance

1. A sequence is a set of numbers arranged in a particular order by some assigned law.
2. nth term of a sequence is called its general term and is generally denoted by a_n or t_n.
3. **Arithmetic Progression (A.P.)**

   A sequence $a_1, a_2, a_3, \ldots\ldots, a_n$ is said to be an A.P. if

   $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots\ldots = d$ (say)

   i.e., if **$a_n - a_{n-1} = d$, a constant independent of n** for $n = 2, 3, \ldots\ldots$.

   This constant d is called the common difference (c.d.) of A.P.

   The general form of A.P. is $a, a + d, a + 2d, \ldots\ldots$ where a is the first term of A.P.
4. nth term of A.P. $a, a + d, a + 2d, \ldots.$ is

   $a_n (= t_n) = a + (n - 1) d$.

   This is the formula for nth term of A.P. from the beginning.
5. pth term from the end of an A.P. $l, l - d, l - 2d, \ldots\ldots$ is $l - (p - 1) d$, where l is the last term of A.P.

   Remark. It may be noted that $l = t_n$ and $l \neq n$.
6. If each term of an A.P. is (*i*) increased by the same number k (*ii*) decreased by the same number k (*iii*) multiplied by the same number k (*iv*) divided by the same number k; then the resulting sequence is also an A.P. ($k \neq 0$).
7. In an A.P; if the sum of three or four or five numbers is given, then take numbers as:

   (*i*) Three numbers in A.P. are $a - d, a, a + d$.

   (*ii*) Five numbers in A.P. are $a - 2d, a - d, a, a + d, a + 2d$

   (*iii*) Four numbers in A.P. are $a - 3d, a - d, a + d, a + 3d$.
8. Let a be the first term, d be the common difference, l be the last term, n be the number of terms and S_n denotes the sum to first n terms of an A.P.,

then S_n of A.P. = $\frac{n}{2}$ [2a + (n − 1) d]

or S_n of A.P. = $\frac{n}{2}$ (a + l).

9. To find T_n if S_n is given

$T_1 = S_1$ and $T_n = S_n - S_{n-1}$ for all $n \geq 2$

10. A.M. (Arithmetic Mean) between a and b is $\frac{a+b}{2}$.

11. $A_1, A_2, ...\ A_n$ are said to be n A.M.'s between a and b if $a, A_1, A_2, ... A_n, b$ are in A.P. and this A.P. sequence has $(n + 2)$ terms where $T_{n+2} = b$

12. Geometric Progression G.P.

A sequence $a_1, a_2, a_3, ..., a_n$ of n numbers where none of a_n's is zero is said to be a G.P. if $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3}$ $= ...... = r$ (say).

i.e., if $\frac{a_n}{a_{n-1}} = r$, a constant independent of n for $n = 2, 3, ...$.

This constant r is called the **common ratio** (c.r.) of G.P.

The general form of G.P. is $a, ar, ar^2, ...$ where a is the first term of G.P.

13. nth term of G.P., $a, ar, ar^2, ...$ is $a_n\ (= t_n) = ar^{n-1}$.

This is formula for nth term of G.P. from the beginning.

14. pth term from the end of a G.P. $l, \frac{l}{r}, \frac{l}{(r^2)}, ...,$ is $\frac{l}{(r^{p-1})}$

where l is the last term of G.P.

Remark. It may be noted that $l = t_n$ and $l \neq n$.

15. If each term of a G.P. is (*i*) multiplied by the same number k (*ii*) divided by the same number k ($k \neq 0$), then the resulting sequence is also a G.P.

16. Take **three** numbers in G.P. as

(*i*) $\frac{a}{r}, a, ar$ if their product is given,

(*ii*) a, ar, ar^2 if their product is not given.

17. Sum to n terms of a G.P.

Let a be the first term, r be the common ratio, l be the last term, n be the number of terms and S_n denote the sum to n terms of a G.P.; then

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) = \frac{lr - a}{r - 1} \text{ if } r > 1$$

$$= \frac{-a(1 - r^n)}{-(1 - r)} = \frac{a(1 - r^n)}{1 - r} = \frac{a - lr}{1 - r} \text{ if } r < 1$$

18. G.M. between a and b is $\sqrt{ab}$.

19. $G_1, G_2, \ldots\ldots, G_n$ are said to be n G.M.'s between a and b if $a, G_1, G_2, \ldots\ldots G_n, b$ is a G.P. sequence of $(n + 2)$ terms. and $T_{n+2} = b$

20. Rule of finding S_n if t_n is given

Putting $n = 1, 2, 3, \ldots\ldots\ldots, n$ in t_n and adding vertically, we get S_n.

21. If $t_n = an^3 + bn^2 + cn + d$, then $S_n = a\ \Sigma n^3 + b\ \Sigma n^2 + c\ \Sigma n + dn$.

22. $\Sigma n = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$.

23. $\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

24. $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$

TEXTBOOK QUESTIONS SOLVED

EXERCISE 9.1 *(Page No.: 180–181)*

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n th terms are:

1. $a_n = n(n + 2)$

Sol. Here $a_n = n(n + 2)$

Putting $n = 1$, $\quad a_1 = 1(1 + 2) = 3$

Putting $n = 2$, $\quad a_2 = 2(2 + 2) = 8$

Putting $n = 3$, $\quad a_3 = 3(3 + 2) = 15$

Putting $n = 4$, $a_4 = 4(4 + 2) = 24$

Putting $n = 5$, $a_5 = 5(5 + 2) = 35$

Therefore, the required terms are 3, 8, 15, 24 and 35.

2. $a_n = \frac{n}{n+1}$

Sol. Here $a_n = \frac{n}{n+1}$

Putting $n = 1, 2, 3, 4$ and 5, we have

$$a_1 = \frac{1}{1+1} = \frac{1}{2}, \quad a_2 = \frac{2}{2+1} = \frac{2}{3}, \quad a_3 = \frac{3}{3+1} = \frac{3}{4},$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5}, \quad a_5 = \frac{5}{5+1} = \frac{5}{6}$$

Therefore, the required terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$.

3. $a_n = 2^n$

Sol. Here $a_n = 2^n$

Putting $n = 1, 2, 3, 4$ and 5, we have

$a_1 = 2^1 = 2$, $a_2 = 2^2 = 4$, $a_3 = 2^3 = 8$,

$a_4 = 2^4 = 16$, $a_5 = 2^5 = 32$

Therefore, the required terms are 2, 4, 8, 16, 32.

4. $a_n = \frac{2n-3}{6}$

Sol. Here $a_n = \frac{2n-3}{6}$

Putting $n = 1, 2, 3, 4$ and 5, we have

$$a_1 = \frac{2\times1-3}{6} = -\frac{1}{6}, \qquad a_2 = \frac{2\times2-3}{6} = \frac{1}{6},$$

$$a_3 = \frac{2\times3-3}{6} = \frac{3}{6} = \frac{1}{2}, \qquad a_4 = \frac{2\times4-3}{6} = \frac{5}{6},$$

$$a_5 = \frac{2\times5-3}{6} = \frac{7}{6}$$

Therefore, the required terms are $-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}$.

5. $a_n = (-1)^{n-1} 5^{n+1}$

Sol. Here $a_n = (-1)^{n-1} . 5^{n+1}$

Putting $n = 1, 2, 3, 4$ and 5, we have

$a_1 = (-1)^0 . 5^2 = 25,$ $a_2 = (-1)^1 . 5^3 = -125,$

$a_3 = (-1)^2 . 5^4 = 625,$ $a_4 = (-1)^3 . 5^5 = -3125,$

$a_5 = (-1)^4 . 5^6 = 15625$

Therefore, the required terms are 25, – 125, 625, – 3125, 15625.

6. $a_n = n\frac{n^2+5}{4}$.

Sol. Here $a_n = \frac{n(n^2+5)}{4}$

Putting $n = 1, 2, 3, 4$ and 5, we have

$a_1 = \frac{1(1^2+5)}{4} = \frac{6}{4} = \frac{3}{2},$ $a_2 = \frac{2(2^2+5)}{4} = \frac{2\times 9}{4} = \frac{9}{2},$

$a_3 = \frac{3(3^2+5)}{4} = \frac{3\times 14}{4} = \frac{21}{2},$

$a_4 = \frac{4(4^2+5)}{4} = \frac{4\times 21}{4} = 21,$

$a_5 = \frac{5(5^2+5)}{4} = \frac{5\times 30}{4} = \frac{75}{2}$

Therefore, the required terms are $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}, 21, \frac{75}{2}$.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose *n* th terms are:

7. $a_n = 4n - 3; a_{17}, a_{24}$

Sol. Here $a_n = 4n - 3$

Putting $n = 17$, we have $a_{17} = 4 \times 17 - 3 = 65$

Putting $n = 24$, we have $a_{24} = 4 \times 24 - 3 = 93$.

8. $a_n = \frac{n^2}{2^n}; a_7$

Sol. Here $a_n = \frac{n^2}{2^n}$

Putting $n = 7$, we have $a_7 = \frac{7^2}{2^7} = \frac{49}{128}$.

9. $a_n = (-1)^{n-1} n^3; a_9$

Sol. Here $a_n = (-1)^{n-1} . n^3$

Putting $n = 9$, we have $a_9 = (-1)^8 . 9^3$

$= (+1)\ 729 = 729.$

10. $a_n = \frac{n(n-2)}{n+3}; a_{20}$

Sol. Here $a_n = \frac{n(n-2)}{n+3}$

Putting $n = 20$, we have $a_{20} = \frac{20 \times 18}{23} = \frac{360}{23}$.

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3, a_n = 3a_{n-1} + 2$ **for all** $n > 1$.

Sol. Here $a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$.

Putting $n = 2, a_2 = 3a_1 + 2 = 3 \times 3 + 2 = 11$

Putting $n = 3, a_3 = 3a_2 + 2 = 3 \times 11 + 2 = 35$

Putting $n = 4, a_4 = 3a_3 + 2 = 3 \times 35 + 2 = 107$

Putting $n = 5, a_5 = 3a_4 + 2 = 3 \times 107 + 2 = 323$

Therefore, the first five terms of the sequence are 3, 11, 35, 107, 323.

$\therefore$ Corresponding series is $3 + 11 + 35 + 107 + 323 + \ldots$.

12. $a_1 = -1, a_n = \frac{a_{n-1}}{n}, n \geq 2$

Sol. Here $a_1 = -1, a_n = \frac{a_{n-1}}{n}, n \geq 2$

Putting $n = 2, a_2 = \frac{a_1}{2} = \frac{-1}{2}$

Putting $n = 3, a_3 = \frac{a_2}{3} = \frac{-1/2}{3} = -\frac{1}{6}$

Putting $n = 4, a_4 = \frac{a_3}{4} = \frac{-1/6}{4} = -\frac{1}{24}$

Putting $n = 5$, $a_5 = \frac{a_4}{5} = \frac{-1/24}{5} = -\frac{1}{120}$

Therefore, the first five terms of the sequence are

$-1, -\frac{1}{2}, -\frac{1}{6}, -\frac{1}{24}, -\frac{1}{120}$.

$\therefore$ Corresponding series is

$$-1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right) + \ldots .$$

13. $a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$, $n > 2$

Sol. Here $a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$, $n > 2$

Putting $n = 3$, $a_3 = a_2 - 1 = 2 - 1 = 1$

Putting $n = 4$, $a_4 = a_3 - 1 = 1 - 1 = 0$

Putting $n = 5$, $a_5 = a_4 - 1 = 0 - 1 = -1$

Therefore, the first five terms of the sequence are 2, 2, 1, 0, – 1.

Corresponding series is $2 + 2 + 1 + 0 + (-1) + \ldots$.

14. The Fibonacci sequence is defined by $1 = a_1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$, $n > 2$.

Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$.

Sol. Here $a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$, $n > 2$

Putting $n = 3$, $a_3 = a_2 + a_1 = 1 + 1 = 2$

Putting $n = 4$, $a_4 = a_3 + a_2 = 2 + 1 = 3$

Putting $n = 5$, $a_5 = a_4 + a_3 = 3 + 2 = 5$

Putting $n = 6$, $a_6 = a_5 + a_4 = 5 + 3 = 8$

For $n = 1$, $\frac{a_{n+1}}{a_n} = \frac{a_2}{a_1} = \frac{1}{1} = 1$

For $n = 2$, $\frac{a_{n+1}}{a_n} = \frac{a_3}{a_2} = \frac{2}{1} = 2$

For $n = 3$, $\frac{a_{n+1}}{a_n} = \frac{a_4}{a_3} = \frac{3}{2}$

For $n = 4$, $\frac{a_{n+1}}{a_n} = \frac{a_5}{a_4} = \frac{5}{3}$

For $n = 5$, $\frac{a_{n+1}}{a_n} = \frac{a_6}{a_5} = \frac{8}{5}$.

EXERCISE 9.2 *(Page No.: 185–186)*

1. Find the sum of odd integers from 1 to 2001.

Sol. Required sum = 1 + 3 + 5 + ... + 2001.

(From 1 to 2001 ⇒ Both 1 and 2001 are included)

It is an arithmetic series with $a = 1$, $d = 3 - 1 = 5 - 3 = 2$

Let $a_n = 2001$

then $1 + (n - 1) \times 2 = 2001$

[$\because a_n$ (or t_n) of A.P. $= a + (n-1)d$]

$\Rightarrow (n - 1) \times 2 = 2000$

$\Rightarrow n - 1 = 1000$

$\Rightarrow n = 1001$

$\therefore$ Required sum (of A.P.) $= \frac{n}{2}(a + l) = \frac{1001}{2}(1 + 2001)$

$= 1001 \times 1001 = 1002001.$

2. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.

Sol. Required sum = 105 + 110 + 115 + ... + 995

(**Between** 100 and 1000 ⇒ both 100 and 1000 are excluded)

It is an arithmetic series with $a = 105$, $d = 5$.

Let $a_n = 995$ then $105 + (n - 1) \times 5 = 995$

$\Rightarrow (n - 1) \times 5 = 890$ $\Rightarrow n - 1 = \frac{890}{4} = 178$

$\Rightarrow n = 179$

$\therefore$ Required sum $= \frac{n}{2}(a + l) = \frac{179}{2}(105 + 995)$

$= \frac{179}{2} \times 1100 = 179 \times 550 = 98450.$

3. In an A.P., the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20^{th} term is – 112.

Sol. Here $a = 2$. Let d be the common difference of A.P.

By the given condition,

sum of first five terms = $\frac{1}{4}$(sum of next five terms)

i.e., $t_1 + t_2 + t_3 + t_4 + t_5 = \frac{1}{4}(t_6 + t_7 + t_8 + t_9 + t_{10})$

i.e., $a + a + d + a + 2d + a + 3d + a + 4d$

$$= \frac{1}{4}[a + 5d + a + 6d + a + 7d + a + 8d + a + 9d]$$

or $5a + 10d = \frac{1}{4}[5a + 35d]$

cross-multiplying,

or $20a + 40d = 5a + 35d$

or $5d = -15a = -15(2) = -30$

$\therefore$ $d = -6$

$\therefore$ t_{20} of A.P. $= a + (20 - 1)d = a + 19d$

$= 2 + 19(-6) = 2 - 114 = -112.$

4. How many terms of the A.P. $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum -25?

Sol. Here $a = -6$, $d = -\frac{11}{2} - (-6) = \frac{1}{2}$

Let -25 be the sum of n terms of this A.P. ($n \in N$)

Using $S_n = \frac{n}{2}[2a + (n-1)d]$, we have

$$-25 = \frac{n}{2}\left[2(-6) + (n-1)\left(\frac{1}{2}\right)\right]$$

or $-50 = n\left[-12 + \left(\frac{n-1}{2}\right)\right] = n\left(\frac{-24 + n - 1}{2}\right)$

or $-50 = n\left(\frac{n-25}{2}\right)$

or $-100 = n^2 - 25n$ or $n^2 - 25n + 100 = 0$

or $n^2 - 5n - 20n + 100 = 0 \Rightarrow n(n-5) - 20(n-5) = 0$

or $(n-5)(n-20) = 0$ $\therefore$ $n = 5, 20.$

Both the values of n are natural numbers, and therefore, admissible.

Explanation of Double Answer

$$S_{20} = -6 - \frac{11}{2} - 5 - \frac{9}{2} - 4 - \frac{7}{2} - 3 - \frac{5}{2} - 2 - \frac{3}{2}$$
$$- 1 - \frac{1}{2} + 0 + \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2}$$
$$= -6 - \frac{11}{2} - 5 - \frac{9}{2} - 4 \text{ (other terms cancel)}$$
$$= S_5.$$

5. In an A.P., if p^{th} term is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of first pq terms is $\frac{1}{2}(pq + 1)$, where $p \neq q$.

Sol. Let a be the first term and d, the C.D. of A.P.

$$a_p = \frac{1}{q} \quad \Rightarrow \quad a + (p-1)\,d = \frac{1}{q} \quad \text{...(i)}$$

$$a_q = \frac{1}{p} \quad \Rightarrow \quad a + (q-1)\,d = \frac{1}{p} \quad \text{...(ii)}$$

Subtracting (*ii*) from (*i*), (to eliminate a) we have

$$(p-1)\,d - (q-1)\,d = \frac{1}{q} - \frac{1}{p}$$

$$\Rightarrow \quad pd - d - qd + d = \frac{1}{q} - \frac{1}{p}$$

$$(p-q)\,d = \frac{1}{q} - \frac{1}{p}$$

$$\Rightarrow \quad (p-q)\,d = \frac{p-q}{pq}$$

Dividing by $p - q$, which is non-zero since $p \neq q$,

$$d = \frac{1}{pq}$$

Putting this value of d in (*i*), we get

$$a + (p-1)\,.\,\frac{1}{pq} = \frac{1}{q}$$

$\Rightarrow \quad a + \frac{1}{q} - \frac{1}{pq} = \frac{1}{q} \quad \Rightarrow \quad a = \frac{1}{pq}$

Putting these values of $a = \frac{1}{pq}$, $d = \frac{1}{pq}$ and $n = pq$ in

$S_n = \frac{n}{2}[\,2a + (n - 1)\,d]$, we have

$$S_{pq} = \frac{pq}{2}\left[2\times\frac{1}{pq} + (pq-1)\times\frac{1}{pq}\right]$$

$$= \frac{pq}{2}\left[\frac{2}{pq} + \frac{pq-1}{pq}\right] = \frac{pq}{2} \cdot \frac{2+pq-1}{pq} = \frac{1}{2}(pq + 1).$$

6. If the sum of a certain number of terms of the A.P. 25, 22, 19, ..., is 116, find the last term.

Sol. The given A.P. is 25, 22, 19, ...

Here $a = 25$, $d = -3$ (= 22 – 25)

Let 116 be the sum of n terms, then

$$S_n = 116$$

$$\Rightarrow \quad \frac{n}{2}[2 \times 25 + (n-1)(-3)] = 116$$

$$\Rightarrow \quad \frac{n}{2}[50 - 3n + 3] = 116$$

$$\Rightarrow \quad \frac{n}{2}(53 - 3n) = 116$$

$$\Rightarrow \quad 53n - 3n^2 = 232$$

$$\Rightarrow \quad 3n^2 - 53n + 232 = 0$$

Solving by quadratic formula, here

$$a = 3,\ b = -53,\ c = 232$$

$$D = b^2 - 4ac = (-53)^2 - 4(3)(232)$$

$$= 2809 - 2784 = 25$$

$$\therefore \quad n = \frac{-b \pm \sqrt{D}}{2a} = \frac{53 \pm 5}{6} = \frac{29}{3} \text{ or } 8$$

Since n is the number of terms, $n \in N$.

$\therefore$ Rejecting $n = \frac{29}{3}$, we have $n = 8$.

$\therefore$ Last term $= a_8 = a + 7d$
$= 25 + 7(-3) = 4.$

7. Find the sum to n terms of the A.P. whose k^{th} term is $5k + 1$.

Sol. Here $a_k = 5k + 1$

Putting $k = 1$ and $k = n$,

$$a_1 = 5 \times 1 + 1 = 6, \; a_n = 5n + 1$$

$$\therefore \quad S_n = \frac{n}{2}(a_1 + a_n) \quad \left| \frac{n}{2}(a + l) \right.$$

$$= \frac{n}{2}(6 + 5n + 1) = \frac{n}{2}(5n + 7).$$

8. If the sum of n terms of an A.P. is $(pn + qn^2)$, where p and q are constants, find the common difference.

Sol. Here $S_n = pn + qn^2$ ...(*i*)

Putting $n = 1$ in (*i*), we have

$S_1 = p + q$ or $a_1 = p + q$ ...(*ii*)

Putting $n = 2$ in (*i*), we have

$S_2 = 2p + 4q$ or $a_1 + a_2 = 2p + 4q$ ...(*iii*)

Subtracting (*ii*) from (*iii*), $a_2 = p + 3q$

$\therefore$ Common difference $d = a_2 - a_1$
$= (p + 3q) - (p + q) = 2q.$

9. The sums of n terms of two arithmetic progressions are in the ratio $5n + 4 : 9n + 6$. Find the ratio of their 18^{th} terms.

Sol. Let a_1, a_2 be the first terms and d_1, d_2 be the common differences of the first and second arithmetic progressions respectively.

According to the given condition, we have

$$\frac{\text{Sum of } n \text{ terms of first A.P.}}{\text{Sum of } n \text{ terms of second A.P.}} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{\frac{n}{2}[2a_1 + (n-1)\,d_1]}{\frac{n}{2}[2a_2 + (n-1)\,d_2]} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{5n+4}{9n+6} \quad ...(i)$$

Now, $\dfrac{18^{th} \text{ term of first A.P.}}{18^{th} \text{ term of second A.P.}} = \dfrac{a_1 + 17d_1}{a_2 + 17d_2}$

Multiplying both Numerator and denominator by 2,

$$= \frac{2a_1 + 34d_1}{2a_2 + 34d_2} \quad ... (ii)$$

Comparing it with L.H.S. of (i), $n - 1 = 34$ so that $n = 35$.

Putting $n = 35$ on both sides of (i), we have

$$\frac{2a_1 + 34d_1}{2a_2 + 34d_2} = \frac{5 \times 35 + 4}{9 \times 35 + 6} = \frac{179}{321}$$

$\therefore$ From (ii), $\dfrac{18^{th} \text{ term of first A.P.}}{18^{th} \text{ term of second A.P.}} = \dfrac{179}{321}$.

10. If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first $(p + q)$ terms.

Sol. Let a be the first term and d be the common difference of A.P.

Given: $S_p = S_q \quad (p \neq q)$

$\Rightarrow \quad \dfrac{p}{2}[2a + (p - 1)d] = \dfrac{q}{2}[2a + (q - 1)d]$

$\Rightarrow \quad p[2a + (p - 1)d] = q[2a + (q - 1)d]$

$\Rightarrow \quad 2pa + (p^2 - p)d = 2qa + (q^2 - q)d$

$\Rightarrow \quad 2ap + p^2d - pd - 2aq - q^2d + qd = 0$

$\Rightarrow \quad 2(p - q)a + [p^2 - q^2 - p + q]d = 0$

$\Rightarrow \quad 2(p - q)a + [(p + q)(p - q) - (p - q)]d = 0$

$\Rightarrow \quad (p - q)[2a + (p + q - 1)d] = 0$

Dividing by $p - q$, we have

$$2a + (p + q - 1)d = 0 \quad ...(i)$$

Now, $S_{p+q} = \dfrac{p+q}{2}[2a + (p + q - 1)d]$

$= \dfrac{p+q}{2}(0)$ [Using (i)]

$= 0.$

11. Sum of the first p, q and r terms of an A.P. are a, b and c, respectively. Prove that

$$\frac{a}{p}(q - r) + \frac{b}{q}(r - p) + \frac{c}{r}(p - q) = 0.$$

Sol. Let A be the first term and d, the common difference.

$$S_p \text{ of A.P.} = a \quad \therefore \quad \frac{p}{2}[2A + (p-1)d] = a$$

Dividing by p, $\quad \frac{1}{2}[2A + (p-1)d] = \frac{a}{p}$

or $\quad A + \frac{1}{2}(p-1)d = \frac{a}{p}$

Similarly, $\quad A + \frac{1}{2}(q-1)d = \frac{b}{q}$

and $\quad A + \frac{1}{2}(r-1)d = \frac{c}{r}$

Putting these values of $\frac{a}{p}, \frac{b}{q}, \frac{c}{r}$ in

L.H.S. $= \frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q)$, we have

$$\text{L.H.S.} = [A + \frac{1}{2}(p-1)d](q-r) + [A + \frac{1}{2}(q-1)d](r-p) + [A + \frac{1}{2}(r-1)d](p-q)$$

$$= A(q-r) + \frac{1}{2}d(p-1)(q-r) + A(r-p) + \frac{1}{2}d(q-1)(r-p) + A(p-q) + \frac{1}{2}d(r-1)(p-q)$$

$$= A(q-r+r-p+p-q) + \frac{1}{2}d[(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)]$$

$$= A(0) + \frac{1}{2}d(pq - pr - q + r + qr - pq - r + p + pr - qr - p + q)$$

$$= A(0) + \frac{1}{2}d(0) = 0 = \text{R.H.S.}$$

12. The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of m^{th} and n^{th} term is $(2m - 1) : (2n - 1)$.

Sol. Let a be the first term and d, the common difference of A.P.

By the given condition, $\dfrac{S_m}{S_n} = \dfrac{m^2}{n^2}$

$$\Rightarrow \quad \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2}$$

Dividing both sides by $\dfrac{m}{n}$,

$$\frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n} \Rightarrow \frac{2a+md-d}{2a+nd-d} = \frac{m}{n}$$

Cross-multiplying, $2an + mnd - nd = 2am + mnd - md$

or $\quad 2an - 2am = (nd - md)$

or $\quad 2a(n-m) = (n-m)d$

Dividing both sides by $(n-m) \neq 0$,

$$d = 2a \quad ...(i)$$

Now, $\quad \dfrac{t_m}{t_n} = \dfrac{a+(m-1)d}{a+(n-1)d}$

Putting $\quad d = 2a$ from (i),

$$= \frac{a+(m-1).2a}{a+(n-1).2a} = \frac{a(1+2m-2)}{a(1+2n-2)} = \frac{2m-1}{2n-1}.$$

13. If the sum of n terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of m.

Sol. Here $\quad S_n = 3n^2 + 5n \quad ...(i)$

Putting $n = 1$ in (i), $S_1 = 3 + 5$ or $a_1 = 8$

Putting $n = 2$ in (i), $S_2 = 3 \times 4 + 5 \times 2 = 22$

$\Rightarrow a_1 + a_2 = 22 \quad \therefore a_2 = 22 - a_1 = 22 - 8 = 14$

$\therefore a = a_1 = 8, \quad d = a_2 - a_1 = 14 - 8 = 6$

Since $a_m = 164 \quad \therefore a + (m-1)d = 164$

$\Rightarrow \quad 8 + (m-1) \times 6 = 164$

$\Rightarrow 8 + 6m - 6 = 164 \Rightarrow 6m + 2 = 164$ or $6m = 162$

$$\therefore \quad m = \frac{162}{6} = 27.$$

14. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P.

Sol. Let A_1, A_2, A_3, A_4 and A_5 be the five numbers between 8 and 26 such that

8, A_1, A_2, A_3, A_4, A_5, 26 are in A.P.

It is an A.P. sequence of $n + 2 = 5 + 2 = 7$ terms.

Here $a = 8, b = 26, n = 7$

$\therefore \quad a_7 = 26 \quad \Rightarrow \quad 8 + 6d = 26$

$\Rightarrow \quad 6d = 26 - 8 = 18 \Rightarrow d = \dfrac{18}{6} = 3$

Thus, $A_1 = t_2 = a + d = 8 + 3 = 11$

$A_2 = t_3 = a + 2d = 8 + 2 \times 3 = 14$

$A_3 = a + 3d = 8 + 3 \times 3 = 17$

$A_4 = a + 4d = 8 + 4 \times 3 = 20$

$A_5 = t_6 = a + 5d = 8 + 5 \times 3 = 23$

Hence, the required five numbers are 11, 14, 17, 20, 23.

Note: $A_i = a + id$

15. If $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the A.M. between a and b, then find the value of n.

Sol. A.M. between a and $b = \dfrac{a+b}{2}$

Given: $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}} = \dfrac{a+b}{2}$

Cross-multiplying; $2(a^n + b^n) = (a + b)(a^{n-1} + b^{n-1})$

$\Rightarrow \quad 2a^n + 2b^n = a(a^{n-1} + b^{n-1}) + b(a^{n-1} + b^{n-1})$

$\Rightarrow \quad 2a^n + 2b^n = a^n + ab^{n-1} + ba^{n-1} + b^n$

$\Rightarrow \quad 2a^n - a^n + 2b^n - b^n = ab^{n-1} + ba^{n-1}$

$\Rightarrow \quad a^n + b^n = ab^{n-1} + ba^{n-1}$

$\Rightarrow \quad a^n - ba^{n-1} = ab^{n-1} - b^n$

$\Rightarrow \quad a^{n-1}(a - b) = b^{n-1}(a - b)$

[$\because a^n = a^{n-1+1} = a^{n-1}a$. Similarly $b^n = b^{n-1}b$]

Dividing by $a - b$; $\quad a^{n-1} = b^{n-1} \Rightarrow \dfrac{a^{n-1}}{b^{n-1}} = 1$

$\Rightarrow \quad \left(\dfrac{a}{b}\right)^{n-1} = \left(\dfrac{a}{b}\right)^0 \Rightarrow n - 1 = 0$

$\therefore \quad n = 1.$

16. Between 1 and 31, *m* numbers have been inserted in such a way that the resulting sequence is an A.P. and the ratio of 7th and (*m* – 1)th numbers is 5 : 9. Find the value of *m*.

Sol. Let $A_1, A_2, A_3, ..., A_m$ be the m numbers between 1 and 31 such that

$1, A_1, A_2, A_3, ..., A_m, 31$ are in A.P.

Here $a = 1, \quad b = 31, n = m + 2$

$\therefore \quad a_{m+2} = 31 \Rightarrow 1 + (m + 2 - 1)d = 31$

$\Rightarrow 1 + (m + 1)d = 31$

$$\Rightarrow \quad d = \frac{30}{m+1}$$

Given: $\dfrac{A_7}{A_{m-1}} = \dfrac{5}{9}$

(We know that $Ai = a + id$. See Note Q.No. 14)

$$\Rightarrow \quad \frac{a+7d}{a+(m-1)d} = \frac{5}{9} \quad \Rightarrow \quad 9a + 63d = 5a + 5(m-1)d$$

$$\Rightarrow \quad 4a = (5m - 5 - 63)d$$

Putting values of a and d

$$\Rightarrow \quad 4 \times 1 = (5m - 68) \times \frac{30}{m+1}$$

$$\Rightarrow \quad 4(m + 1) = 30(5m - 68)$$

$$\Rightarrow \quad 4m + 4 = 150m - 2040$$

$$\Rightarrow \quad -146m = -2044$$

$$\Rightarrow \quad m = \frac{2044}{146} = \frac{1022}{73} = 14.$$

17. A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalment by ₹ 5 every month, what amount he will pay in the 30th instalment?

Sol. The instalments for 1st, 2nd, 3rd, ..., months, in rupees, are 100, 105, 110, ... ($\because$ Increase in each instalment is ₹ 5)

They form an A.P. with $a = 100$, $d = 5$

(105 – 100 = 110 – 105)

Amount of 30th instalment = the 30th term of this A.P.

$= a_{30} = a + 29d = 100 + 29 \times 5$

$= 100 + 145 = 245$

$\therefore$ Amount of 30th instalment = 245.

18. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

Sol. Let n be the number of sides of the polygon.

We know that sum of all the angles of a n sided figure

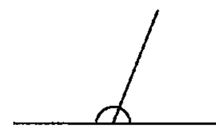
$= n \times 180°$ (By theorem of linear pair)

We also know that sum of exterior angles of a figure = 360°

$\therefore$ Sum of all interior angles

$$= n \times 180° - 360° = (n-2)180° \quad ...(i)$$

Again because interior angles of a polygon are in A.P. with smallest angle $a = 120°$ and common difference $d = 5°$ (given), therefore, sum of all interior angles of the polygon



$$= \frac{n}{2}[2a + (n-1)d] \quad = \frac{n}{2}[240° + (n-1)5°]$$

$$= \frac{n}{2}[240 + 5n - 5]° \quad = \frac{n}{2}(5n + 235)° = \frac{5n}{2}(n+47)° \quad ...(ii)$$

From (*i*) and (*ii*) equating the two values of sum of all the interior angles, we have

$$(n-2)180 = \frac{5n}{2}(n+47)$$

Dividing both sides by 5 and multiplying by 2,

$$(n-2)72 = n(n+47)$$

or $72n - 144 = n^2 + 47n$

or $n^2 - 25n + 144 = 0$

or $n^2 - 9n - 16n + 144 = 0$

or $(n-9)(n-16) = 0$

$\therefore$ $n = 9$ or $n = 16$

But $n = 16$ is impossible because for $n = 16$,

largest angle $= a + (n-1)d = 120° + (16-1)5°$

$= 120° + 75° = 195°$,

which is impossible because no angle in a polygon can exceed 180°. Hence, number of sides in the polygon is $n = 9$.

EXERCISE 9.3 (*Page No.: 192–193*)

1. Find the 20th and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, ...$

Sol. Here $a = \frac{5}{2}$, $r = \frac{5/4}{5/2} = \frac{1}{2}$

We know that a_n of G.P. $= a\,r^{n-1}$

$$\therefore \quad a_{20} = ar^{19} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{2} \times \frac{1}{2^{19}} = \frac{5}{2^{20}}$$

$$a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{2} \times \frac{1}{2^{n-1}} = \frac{5}{2^n}.$$

2. Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

Sol. Here $r = 2$. Let a be the first term.

Given: $a_8 = 192 \Rightarrow ar^7 = 192$

$\Rightarrow a \times 2^7 = 192 \quad \therefore \quad a = \frac{192}{128} = \frac{3}{2}$

$$\therefore \quad a_{12} = ar^{11} = \frac{3}{2} \times 2^{11} = 3 \times 2^{10}$$

$= 3 \times 1024 = 3072.$

3. The 5th, 8th and 11th terms of a G.P. are p, q and s respectively. Show that $q^2 = ps$.

Sol. $a_5 = p \Rightarrow ar^4 = p$

$a_8 = q \Rightarrow ar^7 = q$

$a_{11} = s \Rightarrow ar^{10} = s$

Now, L.H.S. $q^2 = (ar^7)^2 = a^2r^{14}$ and R.H.S. $ps = ar^4 . ar^{10} = a^2r^{14}$

$\therefore$ L.H.S. = R.H.S. *i.e.* $q^2 = ps$.

4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term.

Sol. Here $a = -3$. Let r be the common ratio.

Given: $a_4 = (a_2)^2$

$\Rightarrow ar^3 = (ar)^2 \Rightarrow ar^3 = a^2r^2$

Dividing both sides by ar^2,

$r = a = -3$

$\therefore \quad a_7 = ar^6 = (-3) \times (-3)^6 = -3 \times 729$

$= -2187.$

5. Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4, \ldots,$ **is 128?**

(b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots,$ **is 729?**

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots,$ **is** $\frac{1}{19683}$**?**

Sol. (a) Here the given sequence is a G.P.

$$\left(\because \frac{\text{II}}{\text{I}} = \frac{2\sqrt{2}}{2} = \sqrt{2} \text{ and } \frac{\text{III}}{\text{II}} = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}\right)$$

$$a = 2, r = \sqrt{2}$$

Let $a_n = 128$, then $ar^{n-1} = 128$

$$\Rightarrow \quad 2 \times (\sqrt{2})^{n-1} = 128$$

$$\Rightarrow \quad (\sqrt{2})^{n-1} = 64 = 2^6 = [(\sqrt{2})^2]^6 = (\sqrt{2})^{12}$$

$$\Rightarrow \quad n - 1 = 12$$

$$\Rightarrow \quad n = 13$$

$\therefore$ 128 is the 13^{th} term of the G.P.

(b) Here the given sequence is a G.P.

$$\left(\because \frac{\text{II}}{\text{I}} = \frac{3}{\sqrt{3}} = \sqrt{3} \text{ and } \frac{\text{III}}{\text{II}} = \frac{3\sqrt{3}}{3} = \sqrt{3}\right)$$

$$a = \sqrt{3}, \quad r = \sqrt{3}$$

Let $a_n = 729$, then $ar^{n-1} = 729$

$$\Rightarrow \quad \sqrt{3} \times (\sqrt{3})^{n-1} = 729 \Rightarrow (\sqrt{3})^n = 3^6 = (\sqrt{3})^{12}$$

$$\Rightarrow \quad n = 12$$

$\therefore$ 729 is the 12^{th} term of the G.P.

(c) Here the given sequence is a G.P.

$$\left(\because \frac{\text{II}}{\text{I}} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3} \text{ and } \frac{\text{III}}{\text{II}} = \frac{\frac{1}{27}}{\frac{1}{9}} = \frac{1}{3}\right)$$

$$a = \frac{1}{3}, r = \frac{1}{3}$$

Let $a_n = \frac{1}{19683}$, then $\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683}$

$\Rightarrow \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^9 \quad \Rightarrow n = 9$

$\therefore \frac{1}{19683}$ is the 9th term of the G.P.

3	19683
3	6561
3	2187
3	729
3	243
3	81
3	27
3	9
	3

6. For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Sol. $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P. $\therefore \frac{\text{II}}{\text{I}} = \frac{\text{III}}{\text{II}}$

$\Rightarrow \frac{x}{-\frac{2}{7}} = \frac{-\frac{7}{2}}{x}$ Cross-multiplying $x^2 = \left(-\frac{2}{7}\right)\left(-\frac{7}{2}\right) = 1$

$\therefore \quad x = \pm 1.$

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ..., 20 terms.

Sol. Here $a = 0.15 = \frac{15}{100}$, $r = \frac{0.015}{0.15} = \frac{\frac{15}{1000}}{\frac{15}{100}} = \frac{1}{10} < 1$, $n = 20$

$$\therefore \quad S_{20} = \frac{a(1-r^{20})}{1-r} = \frac{\frac{15}{100}\left[1-\left(\frac{1}{10}\right)^{20}\right]}{1-\frac{1}{10}=\frac{9}{10}}$$

$$= \frac{10}{9} \times \frac{3}{20}\,[1-(0.1)^{20}] = \frac{1}{6}\,[1-(0.1)^{20}].$$

8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, ..., n$ terms.

Sol. Here $a = \sqrt{7}$, $r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3} > 1$

$$\therefore \quad S_n = \frac{a(r^n-1)}{r-1} = \frac{\sqrt{7}\,[(\sqrt{3})^n - 1]}{\sqrt{3}-1}$$

Rationalising

$$= \frac{\sqrt{7}(\sqrt{3}+1)[(\sqrt{3})^n - 1]}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{\sqrt{7}(\sqrt{3}+1)[(3^{1/2})^n - 1]}{3-1}$$

$$= \frac{\sqrt{7}}{2}(\sqrt{3}+1)(3^{n/2}-1).$$

9. 1, – *a*, a^2, – a^3, ..., *n* terms (if $a \neq -1$).

Sol. Here $a_1 = 1$, $r = -a$

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{1[1-(-a)^n]}{1-(-a)}$$

$$= \frac{1-(-a)^n}{1+a}.$$

Remark: Here for $r = -a$, we don't know whether $r < 1$ or r>1. Therefore the reader at his/her will can also use

$$S_n = \frac{a(r^n - 1)}{r-1}$$

10. x^3, x^5, x^7, ..., *n* terms (if $x \neq \pm 1$).

Sol. Here $a = x^3$, $r = \frac{x^5}{x^3} = x^2$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{x^3[1-(x^2)^n]}{1-x^2} = \frac{x^3(1-x^{2n})}{1-x^2}.$$

11. Evaluate $\sum_{k=1}^{11}(2+3^k)$.

Sol. $\sum_{k=1}^{11}(2+3^k) = (2+3^1) + (2+3^2) + (2+3^3) + ... + (2+3^{11})$

$= (2 + 2 + 2 + ... 11 \text{ times}) + (3 + 3^2 + 3^3$ ... to 11 terms)

$$= 11 \times 2 + \frac{3(3^{11}-1)}{3-1} \quad \Big| \because S_n \text{ of G.P.} = \frac{a(r^n-1)}{r-1} \text{ for } r > 1$$

$$= 22 + \frac{3}{2}(3^{11} - 1).$$

12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Sol. Let the three terms in G.P. be $\frac{a}{r}$, a, ar.

Their product = 1 (given)

$\Rightarrow \quad \frac{a}{r} \cdot a \cdot ar = 1 \quad \Rightarrow \quad a^3 = 1^3 \quad \therefore \quad a = 1$

The sum of terms $= \frac{39}{10}$ (given)

$\Rightarrow \quad \frac{a}{r} + a + ar = \frac{39}{10} \quad \Rightarrow \quad \frac{1}{r} + 1 + r = \frac{39}{10} \quad (\because a = 1)$

$\Rightarrow \quad \frac{1+r+r^2}{r} = \frac{39}{10}$

Cross-multiplying

$10 + 10r + 10r^2 = 39r$

$\Rightarrow \quad 10r^2 - 29r + 10 = 0$

$\Rightarrow \quad 10r^2 - 25r - 4r + 10 = 0$

$\Rightarrow \quad 5r(2r - 5) - 2(2r - 5) = 0$

$\Rightarrow \quad (2r - 5)(5r - 2) = 0$

$\Rightarrow \quad r = \frac{5}{2} \text{ or } \frac{2}{5}$

When $a = 1$, $r = \frac{5}{2}$, the terms are

$\frac{1}{5/2}, 1, 1 \times \frac{5}{2} \quad$ *i.e.,* $\quad \frac{2}{5}, 1, \frac{5}{2}$

When $a = 1$, $r = \frac{2}{5}$, the terms are

$\frac{1}{2/5}, 1, 1 \times \frac{2}{5} \quad$ *i.e.,* $\quad \frac{5}{2}, 1, \frac{2}{5}$.

13. How many terms of G.P. 3, 3^2, 3^3, ..., are needed to give the sum 120?

Sol. Here $a = 3$, $r = \frac{9}{3} = 3 > 1$

Let $S_n = 120$, then $\frac{3(3^n - 1)}{3-1} = 120$

$\Rightarrow \quad \frac{3}{2}(3^n - 1) = 120$

$\Rightarrow \quad 3(3^n - 1) = 240 \Rightarrow (3^n - 1) = \frac{240}{3} = 80$

$\Rightarrow \quad 3^n = 80 + 1 = 81 \qquad \Rightarrow \qquad 3^n = 3^4$

$\Rightarrow \quad n = 4$

$\therefore$ 120 is the sum of 4 terms of G.P.

14. The sum of first three terms of a G.P. is 16 and the sum of next three terms is 128. Determine the first term, the common ratio and the sum to *n* terms of the G.P.

Sol. Let a be the first term and r be the common ratio of G.P.

Given: $a_1 + a_2 + a_3 = 16,$

$\Rightarrow \quad a + ar + ar^2 = 16$

$\Rightarrow \quad a(1 + r + r^2) = 16 \qquad ...(i)$

Also, $a_4 + a_5 + a_6 = 128$

$\Rightarrow \quad ar^3 + ar^4 + ar^5 = 128$

$\Rightarrow \quad ar^3(1 + r + r^2) = 128 \qquad ...(ii)$

Dividing (*ii*) by (*i*) (To eliminate a), we have

$r^3 = 8$ or $r^3 = 2^3 \quad \therefore \quad r = 2 > 1$

Putting $r = 2$ in (*i*), $a(1 + 2 + 4) = 16$

or $\quad 7a = 16 \qquad \therefore \quad a = \frac{16}{7}$

$$S_n = \frac{a(r^n - 1)}{r-1} = \frac{\frac{16}{7}(2^n - 1)}{2-1} = \frac{16}{7}(2^n - 1)$$

Hence, $a = \frac{16}{7}$, $r = 2$, $S_n = \frac{16}{7}(2^n - 1)$.

15. Given a G.P. with a = 729 and 7th term 64, determine S_7.

Sol. Here a = 729. Let r be the common ratio.

Given: $a_7 = 64 \quad \Rightarrow \quad ar^6 = 64$

$[\because a_n \text{ of G.P.} = ar^{n-1}]$

$$\Rightarrow \quad 729\,r^6 = 64 \quad \Rightarrow \quad r^6 = \frac{64}{729} = \left(\pm\frac{2}{3}\right)^6$$

$$\therefore \quad r = \pm\frac{2}{3}$$

When $r = \frac{2}{3}$, $\quad S_7 = \frac{a(1-r^7)}{1-r} = \frac{729\left(1-\frac{128}{2187}\right)}{1-\frac{2}{3}=\frac{1}{3}}$

$$= 3 \times 729 \times \frac{2059}{2187} = 2059$$

When $r = -\frac{2}{3}$, $\quad S_7 = \frac{a(1-r^7)}{1-r} = \frac{729\left[1-\left(-\frac{2}{3}\right)^7\right]}{1-\left(-\frac{2}{3}\right)=\frac{5}{3}}$

$$= \frac{3}{5} \times 729\left[1+\frac{128}{2187}\right]$$

$$= \frac{2187}{5} \times \frac{2315}{2187} = 463.$$

16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term.

Sol. Let a be the first term and r be the common ratio.

Given: $a_1 + a_2 = -4$

$\Rightarrow \quad a + ar = -4 \quad$ or $\quad a(1 + r) = -4 \quad$ ...(*i*)

Also $\quad a_5 = 4a_3$

$\Rightarrow \quad ar^4 = 4ar^2$

Dividing both sides by ar^2,

$\Rightarrow \quad r^2 = 4 \quad$ ($\because$ a and r cannot be zero)

$\therefore \quad r = \pm 2$

When r = 2, from (*i*), $\quad a(1 + 2) = -4$

$\Rightarrow \quad 3a = -4 \quad \Rightarrow \quad a = -\frac{4}{3}$

G.P. is $a, ar, ar^2, \ldots$ i.e., $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$

When $r = -2$, from (*i*), $a(1-2) = -4$

$\Rightarrow \quad -a = -4 \quad \Rightarrow \quad a = 4$

G.P. is 4, – 8, 16, – 32, 64, ...

17. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

Sol. Let a be the first term and r be the common ratio.

Given: $a_4 = x \Rightarrow ar^3 = x$

$a_{10} = y \Rightarrow ar^9 = y$

$a_{16} = z \Rightarrow ar^{15} = z$

Now, $\frac{y}{x} = \frac{ar^9}{ar^3} = r^6$ and $\frac{z}{y} = \frac{ar^{15}}{ar^9} = r^6$

$\therefore \quad \frac{y}{x} = \frac{z}{y} \quad (\because \text{ Each } = r^6)$

$\Rightarrow$ x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888, ...

Sol. 8 + 88 + 888 + ... to n terms [all 8's]

This series is neither an A.P. nor a G.P. but we can relate it to a G.P. as explained below:

Taking 8 common, = 8(1 + 11 + 111 + ... to n terms) [all 1's]

Dividing and multiplying by 9,

$= \frac{8}{9}(9 + 99 + 999 + \ldots \text{ to } n \text{ terms})$ [all 9's]

[If the repeated digit is different from 9, first create all 9's]

$= \frac{8}{9}[(10-1) + (10^2-1) + (10^3-1) + \ldots \text{ to } n \text{ terms}]$

$= \frac{8}{9}[(10 + 10^2 + 10^3 + \ldots \text{ to } n \text{ terms}) - (1 + 1 + \ldots n \text{ terms})]$

Using S_n of G.P.= $\frac{a(r^n-1)}{r-1}$ with $a = 10$, $r = 10 > 1$

$$= \frac{8}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right] = \frac{8}{9}\left[\frac{10^{n+1} - 10}{9} - n\right]$$

$$= \frac{8}{81}(10^{n+1} - 10 - 9n)$$

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

Sol. Required sum = $(2)(128) + (4)(32) + (8)(8) + (16)(2) + (32)\left(\frac{1}{2}\right)$

$$= 256 + 128 + 64 + 32 + 16$$

$$\left(\text{a G.P. with } a = 256, r = \frac{1}{2}, < 1, n = 5\right)$$

$$= \frac{256\left[1 - \left(\frac{1}{2}\right)^5\right]}{1 - \frac{1}{2} = \frac{1}{2}} = 2 \times 256\left(1 - \frac{1}{32}\right)$$

$$= 2 \times 256 \times \frac{31}{32} = 496.$$

20. Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and A, AR, AR^2, ... AR^{n-1} form a G.P., and find the common ratio.

Sol. Products of corresponding terms are

$$(a)(A), (ar)(AR), (ar^2)(AR^2), \ldots, (ar^{n-1})(AR^{n-1})$$

i.e., $aA, aA(rR), aA(rR)^2, \ldots, aA(rR)^{n-1}$ which form a G.P.

with common ratio rR. $\left(\because \frac{\text{II}}{\text{I}} = \frac{aA(rR)}{aA} = rR \text{ and } \frac{\text{III}}{\text{II}} = \frac{aA(rR)^2}{aA(rR)} = rR\right)$

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

Sol. Let the four numbers in G.P. be a, ar, ar^2, ar^3.

Given: $T_3 - T_1 = 9$

$\Rightarrow ar^2 - a = 9$ or $a(r^2 - 1) = 9$ ...(*i*)

Also $T_2 - T_4 = 9 \Rightarrow ar - ar^3 = 18$ or $ar^3 - ar = -18$

or $ar(r^2 - 1) = -18$ ...(*ii*)

Dividing (*ii*) by (*i*), (to eliminate *a*), $r = -2$

Putting in (*i*), $a(4 - 1) = 9$ or $3a = 9$ or $a = 3$.

$\therefore$ The four numbers in G.P. are 3, – 6, 12, – 24.

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are *a*, *b* and *c*, respectively, prove that

$$a^{q-r}\, b^{r-p}\, c^{p-q} = 1.$$

Sol. Let A be the first term and R be the common ratio.

$a_p = a \quad \Rightarrow \quad AR^{p-1} = a$

$a_q = b \quad \Rightarrow \quad AR^{q-1} = b$

$a_r = c \quad \Rightarrow \quad AR^{r-1} = c$

Putting these values of *a*, *b*, *c*, in L.H.S. we have

$a^{q-r}\, b^{r-p}\, c^{p-q} = (AR^{p-1})^{q-r} . (AR^{q-1})^{r-p} (AR^{r-1})^{p-q}$

$= A^{q-r}\, R^{(p-1)(q-r)} . A^{r-p} . R^{(q-1)(r-p)} . A^{p-q} . R^{(r-1)(p-q)}$

$= A^{q-r+r-p+p-q} . R^{pq-pr-q+r+qr-pq-r+p+pr-qr-p+q}$

$= A^0 . R^0 = 1 \times 1 = 1.$

23. If the first and the n^{th} terms of a G.P. are *a* and *b*, respectively, and if P is the product of *n* terms, prove that $P^2 = (ab)^n$.

Sol. Let *r* be the common ratio.

$$P = a(ar)(ar^2) \ldots \left(\frac{b}{r^2}\right)\left(\frac{b}{r}\right)b \qquad \ldots(i)$$

Writing the factors on R.H.S. in reverse order

$$P = b\left(\frac{b}{r}\right)\left(\frac{b}{r^2}\right) \ldots (ar^2)\,(ar)\,a \qquad \ldots(ii)$$

Multiplying (*i*) and (*ii*), we get

$$P^2 = ab\left(ar\,\frac{b}{r}\right)\cdot\left(ar^2\,\frac{b}{r^2}\right) \ldots\ldots\ldots \left(\frac{b}{r^2}\,ar^2\right)\left(\frac{b}{r}\,ar\right)(ba)$$

$P^2 = (ab)(ab)(ab) \ldots (ab)(ab)(ab) = (ab)^n.$

24. Show that the ratio of the sum of first *n* terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Sol. Let *a* be the first term and *r* be the common ratio of G.P.

S_1 = Sum of first n terms

$$= \frac{a(1-r^n)}{1-r} \quad ...(i)$$

S_2 = Sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term

$= ar^n + ar^{n+1} + ... + ar^{2n-1}$

(It is a G.P. with first term = ar^n, common ratio = r and number of terms = n)

$$= \frac{ar^n(1-r^n)}{1-r} \left(\frac{A(1-R^n)}{1-R}\right) \quad ...(ii)$$

Dividing (*i*) by (*ii*), we have

$$\frac{S_1}{S_2} = \frac{a(1-r^n)}{1-r} \times \frac{1-r}{ar^n(1-r^n)} = \frac{1}{r^n}.$$

25. If a, b, c and d are in G.P. show that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

Sol. Let r be the common ratio, then

$$b = ar,\ c = ar^2,\ d = t_4 = ar^3$$

Now L.H.S. $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

Putting values of b,c,d;

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$

$= a^2(1 + r^2 + r^4) \cdot a^2r^2(1 + r^2 + r^4)$

$= a^4r^2(1 + r^2 + r^4)^2 \quad ...(i)$

And R.H.S. $(ab + bc + cd)^2$

$= (a \cdot ar + ar \cdot ar^2 + ar^2 \cdot ar^3)^2$

$= (a^2r + a^2r^3 + a^2r^5)^2$

$= [a^2r(1 + r^2 + r^4)]^2$

$= a^4r^2(1 + r^2 + r^4)^2 \quad ...(ii)$

From (*i*) and (*ii*), we have L.H.S. = R.H.S.

i.e. $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Sol. Let the two numbers be G_1 and G_2 so that 3, G_1, G_2, 81 are in G.P. Now this is a G.P of four terms.

Let r be the common ratio.

$a_4 = 81 \quad \Rightarrow \quad 3r^3 = 81$

$\Rightarrow \quad r^3 = 27 = 3^3 \quad \therefore \quad r = 3$

$G_1 = T_2 = ar = 3 \times 3 = 9$, $G_2 = T_3 = ar^2 = 3 \times 3^2 = 27$.

Note : It may be noted that $G_i = ar^i$

27. Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Sol. G.M. between a and $b = \sqrt{ab}$

Given: $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n} = \sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2}$

Cross-multiplying

$$a^{n+1} + b^{n+1} = a^{1/2} b^{1/2} (a^n + b^n)$$

$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{n+1/2} b^{1/2} + a^{1/2} b^{n+1/2}$

$\Rightarrow \quad a^{n+1} - a^{n+1/2} b^{1/2} = a^{1/2} b^{n+1/2} - b^{n+1}$

$\Rightarrow \quad a^{n+1/2}a^{1/2} - a^{n+1/2} b^{1/2} = a^{1/2} b^{n+1/2} - b^{n+1/2} b^{1/2}$

$[\because a^{n+1} = a^{n+1/2+1/2} = a^{n+1/2} a^{1/2}$ and $b^{n+1} = b^{n+1/2+1/2} = b^{n+1/2} b^{1/2}]$

$\Rightarrow \quad a^{n+1/2} (a^{1/2} - b^{1/2}) = b^{n+1/2} (a^{1/2} - b^{1/2})$

Dividing both sides by $a^{1/2} - b^{1/2}$, we have

$a^{n+1/2} = b^{n+1/2}$

Dividing both sides by $b^{n+1/2}$, $\dfrac{a^{n+\frac{1}{2}}}{b^{n+\frac{1}{2}}} = 1$

$$\Rightarrow \left(\frac{a}{b}\right)^{n+1/2} = \left(\frac{a}{b}\right)^0 \Rightarrow n + \frac{1}{2} = 0 \quad \therefore \quad n = -\frac{1}{2}.$$

28. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$.

Sol. Let the two numbers be a and b, $a > b$.

Given: Sum of two numbers = 6 (Their G.M.)

$\Rightarrow a + b = 6\sqrt{ab}$

$$\Rightarrow \quad a + b = 3 \times 2\sqrt{ab} \quad \Rightarrow \quad \frac{a+b}{2\sqrt{ab}} = \frac{3}{1}$$

By componendo and dividendo, we have

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3+1}{3-1} \Rightarrow \frac{(\sqrt{a})^2+(\sqrt{b})^2+2\sqrt{a}\sqrt{b}}{(\sqrt{a})^2+(\sqrt{b})^2-2\sqrt{a}\sqrt{b}} = \frac{4}{2}$$

(Rule of componendo and Dividendo is if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$)

$$\Rightarrow \qquad \frac{(\sqrt{a}+\sqrt{b})^2}{(\sqrt{a}-\sqrt{b})^2} = \frac{2}{1}$$

Taking square root, $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{2}}{1}$

Again, by componendo and dividendo, we get

$$\frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow \qquad \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1} \Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

Squaring both sides,

$$\frac{a}{b} = \frac{2+1+2\sqrt{2}}{2+1-2\sqrt{2}}$$

$$\Rightarrow \qquad \frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}.$$

29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Sol. Let the two quantities be a and b, a > b.

$$A = \frac{a+b}{2} \qquad \therefore \qquad a+b = 2A \qquad ...(i)$$

$$G = \sqrt{ab} \qquad \therefore \qquad ab = G^2 \qquad ...(ii)$$

Let us solve (*i*) and (*ii*) for a and b

We know that $(a-b)^2 = (a+b)^2 - 4ab$

Putting values from (*i*) and (*ii*),

$(a-b)^2 = 4A^2 - 4G^2 = 4(A^2 - G)^2$

Taking square root, $a - b = 2\sqrt{A^2 - G^2}$ $\quad (\because a > b)$ $\quad$ ...(*iii*)

Adding (*i*) and (*iii*) $2a = 2A + 2\sqrt{A^2 - G^2}$

Dividing by 2, $a = A + \sqrt{(A+G)(A-G)}$ $\quad$ ...(*iv*)

Similarly subtracting (*iii*) from (*i*), $b = A - \sqrt{(A+G)(A-G)}$

...(*v*)

From (*iv*) and (*v*), the two quantities a and b are given by

$A \pm \sqrt{(A+G)(A-G)}$

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?

Sol. Given: Originally *i.e.* at $t = 0$ hours, number of bacteria is 30 and it doubles every hour.

$\therefore$ The number of bacteria at the end of 1st, 2nd, 3rd, ... hours is $30 \times 2, 30 \times 2^2, 30 \times 2^3, ...$

which form a G.P.

$\therefore$ Number of bacteria at the end of

(*i*) 2nd hour $= 30 \times 2^2 = 120$

(*ii*) 4th hour $= 30 \times 2^4 = 480$

(*iii*) nth hour $= 30 \times 2^n$.

31. What will ₹ 500 amount to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Sol. Amount at the end of the 1st year

$$= 500 + \frac{500 \times 10 \times 1}{100} = 500(1 + 0.1)$$

[$\because$ C.I in one year = S.I. for the same year = $\frac{P \times R \times T}{100}$]

= ₹ 500(1.1)

Amount at the end of 2nd year

$$= 500(1.1) + \frac{500 \times 1.1 \times 10 \times 1}{100}$$

= 500(1.1)(1 + 0.1) = ₹ $500(1.1)^2$

Similarly, amount at the end of 3rd year

= ₹ $500(1.1)^3$ and so on.

Thus, amounts at the end of 1st, 2nd, 3rd, ... years are

₹ 500(1.1), ₹ $500(1.1)^2$, ₹ $500(1.1)^3$, ...

They form a G.P.

$\therefore$ Required amount at the end of 10 years

= ₹ $500(1.1)^{10}$.

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the qudratic equation.

Sol. Let the roots of the quadratic equation be α and β.

A.M. = 8 $\Rightarrow$ $\frac{\alpha + \beta}{2} = 8$ $\therefore$ $\alpha + \beta = 16$

G.M. = 5 $\Rightarrow$ $\sqrt{\alpha\beta} = 5$ $\therefore$ $\alpha\beta = 25$

Required quadratic equation is $x^2 - Sx + P = 0$

i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Putting values of $\alpha + \beta$ and $\alpha\beta$,

$x^2 - 16x + 25 = 0$.

EXERCISE 9.4 (*Page No.: 196*)

Find the sum to *n* terms of each of the series in Exercises 1 to 7.

1. 1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + ...

Sol. Here a_n = (n^{th} term of A.P. 1, 2, 3, ...)(n^{th} term of A.P. 2, 3, 4, ...)

$= n(n + 1) = n^2 + n$

[$\because$ n^{th} term of A.P. 1,2,3, ... = 1+ (n–1).1

= 1 + n–1 = n

Also n^{th} term of A.P. 2,3,4, ... $= 2 + (n-1).1$
$= 2 + n - 1 = n + 1]$

$$\therefore S_n = \sum n^2 + \sum n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

[$\because$ If $Tn = an^3 + bn^2 + cn + d$, then
$S_n = a\sum n^3 + b\sum n^2 + c\sum n + nd$]

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right] = \frac{n(n+1)}{2}\left[\frac{2n+4}{3}\right]$$

$$= \frac{n(n+1)}{2}\frac{2(n+2)}{3} = \frac{n(n+1)(n+2)}{3}.$$

2. $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$.

Sol. The given series is 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms

Here a_n = (n^{th} term of A.P. 1, 2, 3, ...)
$\times$ (n^{th} term of A.P. 2, 3, 4, ...)
$\times$ (n^{th} term of A.P. 3, 4, 5, ...)

$= [1 + (n-1).1][2 + (n-1).1][3 + (n-1).1]$

[$\because$ a_n **of A.P.** $= a + (n-1)d$]

$= n(n+1)(n+2)$

Expanding, $= n(n^2 + 3n + 2)$

or $a_n = n^3 + 3n^2 + 2n$

$\therefore S_n = \sum n^3 + 3\sum n^2 + 2\sum n$

$$= \frac{n^2(n+1)^2}{4} + 3.\frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2}$$

Taking $n(n+1)$ common,

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{2n+1}{2} + 1\right]$$

$$= \frac{n(n+1)}{4}[n(n+1) + 2(2n+1) + 4]$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}.$$

$\left[\begin{array}{l}\because n^2 + 5n + 6 \\ = n^2 + 3n + 2n + 6 \\ = (n+3)(n+2)\end{array}\right.$

3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$.

Sol. Here a_n = (n^{th} term of A.P. 3, 5, 7, ...)

(n^{th} term of 1, 2, 3, ...)^2

$$= [3 + (n - 1) \times 2]n^2 = (2n + 1)n^2$$

Expanding, $= 2n^3 + n^2$

$$\therefore S_n = 2\sum n^3 + \sum n^2$$

$$= 2 \cdot \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+2n+1}{3}\right]$$

$$= \frac{n}{6}(n + 1)(3n^2 + 5n + 1).$$

4. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$.

Sol. Here $a_n = \frac{1}{(n^{th} \text{ term of } 1, 2, 3, \dots)(n^{th} \text{ term of } 2, 3, 4, \dots)}$

$$= \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}$$

$$\left[\because \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}\right]$$

$$= \frac{1}{n} - \frac{1}{n+1}$$

Putting n = 1, 2, 3, ..., n, we get

$$a_1 = \frac{1}{1} - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3} - \frac{1}{4}$$

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$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

on adding, the terms in R.H.S. cancel out diagonally,

We have $S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$.

5. $\mathbf{5^2 + 6^2 + 7^2 + ... + 20^2}$.

Sol. Required sum on adding and subtracting $1^2 + 2^2 + 3^2 + 4^2$,

$$= (1^2 + 2^2 + 3^2 + ... + 20^2) - (1^2 + 2^2 + 3^2 + 4^2)$$

$$= \sum_{n=1}^{20} n^2 - \sum_{n=1}^{4} n^2$$

$$= \frac{20(20+1)(2\times 20+1)}{6} - \frac{4(4+1)(2\times 4+1)}{6}$$

$$\left[\because \sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{20\times 21\times 41}{6} - \frac{4\times 5\times 9}{6}$$

$$= 2870 - 30 = 2840.$$

6. $\mathbf{3 \times 8 + 6 \times 11 + 9 \times 14 + ...}$

Sol. Here a_n = (n^{th} term of A.P. 3, 6, 9, ...)(n^{th} term of A.P. 8, 11, 14, ...)

$$= [3 + (n - 1) \times 3][8 + (n - 1) \times 3]$$

$$= 3n(3n + 5) = 9n^2 + 15n \text{ on expanding}$$

$$\therefore \quad S_n = 9 \sum n^2 + 15 \sum n$$

$$= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2}$$

$$= n(n + 1) \left[\frac{3}{2}(2n+1) + \frac{15}{2}\right]$$

Taking $\frac{3}{2}$ common,

$$= \frac{3}{2} n (n + 1)(2n + 1 + 5)$$

$$= \frac{3}{2} n (n + 1)(2n + 6) = \frac{3}{2} n (n + 1) 2 (n + 3)$$

$$= 3n(n + 1)(n + 3).$$

7. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$.

Sol. The given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots\ldots + n$ terms.

$$\therefore a_n = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \Sigma n^2$$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{1}{6} n(2n^2 + 3n + 1) \text{ on expanding}$$

$$\Rightarrow a_n = \frac{2n^3 + 3n^2 + n}{6} = \frac{1}{6}[2n^3 + 3n^2 + n]$$

$$\therefore S_n = \frac{1}{6}[2\,\Sigma n^3 + 3\,\Sigma n^2 + \Sigma n]$$

$$= \frac{1}{6}\left[\frac{2n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2} + \left(\frac{2n+1}{2}\right) + \frac{1}{2}\right]$$

$$= \frac{n(n+1)}{12}[n(n+1) + (2n+1) + 1]$$

$$= \frac{n(n+1)}{12}(n^2 + 3n + 2)$$

$$= \frac{n(n+1)(n+1)(n+2)}{12}$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

$[\because n^2 + 3n + 2 = n^2 + 2n + n + 2 = n(n+2) + (n+2) = (n+1)(n+2)]$

Find the sum to n terms of the series in Exercises 8 to 10 whose n^{th} term is given by

8. $n(n + 1)(n + 4)$.

Sol. Given $a_n = n(n + 1)(n + 4) = n(n^2 + 5n + 4)$

Expanding, $= n^3 + 5n^2 + 4n$

$$\therefore S_n = \Sigma n^3 + 5\Sigma n^2 + 4\Sigma n$$

$$= \frac{n^2(n+1)^2}{4} + 5 \cdot \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2}$$

Taking $n(n + 1)$ common,

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{5(2n+1)}{6} + 2\right]$$

$$= \frac{n(n+1)}{12}\,[3n(n+1) + 10(2n+1) + 24]$$

$$= \frac{n(n+1)}{12}(3n^2 + 23n + 34)$$

$$= \frac{n(n+1)(n+2)(3n+17)}{12}.$$

$$\left[\because 3n^2 + 23n + 34 = 3n^2 + 6n + 17n + 34 \right.$$
$$= 3n(n+2) + 17(n+2)$$
$$\left. = (n+2)(3n+17)\right]$$

9. $n^2 + 2^n$.

Sol. Here $a_n = n^2 + 2^n$

Putting $n = 1, 2, 3, \ldots\ldots., n$, we get

$a_1 = 1^2 + 2^1$

$a_2 = 2^2 + 2^2$

$a_3 = 3^2 + 2^3$

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$a_n = n^2 + 2^n$

Adding columnwise, we get

$$S_n = (1^2 + 2^2 + 3^2 + \ldots\ldots. + n^2) + (2^1 + 2^2 + 2^3 + \ldots\ldots. + 2^n)$$

$= \sum n^2 +$ (sum of n terms of a G.P. with $a = r = 2 > 1$)

$$= \frac{n(n+1)(2n+1)}{6} + \frac{2(2^n - 1)}{2-1}$$

$\left|\ \because S_n \text{ of G.P.} = a\frac{(r^n - 1)}{r-1}\right.$

$$= \frac{n}{6}(n+1)(2n+1) + 2(2^n - 1).$$

10. $(2n - 1)^2$.

Sol. Here $a_n = (2n - 1)^2 = 4n^2 - 4n + 1$ on expanding

$\therefore \quad S_n = 4\sum n^2 - 4\sum n + 1.n$

$= 4 \times \frac{n(n+1)(2n+1)}{6} - 4 \times \frac{n(n+1)}{2} + n$

$= n\left[\frac{2}{3}(n+1)(2n+1) - 2(n+1) + 1\right]$

$= n\left[\frac{2(2n^2+3n+1) - 6(n+1) + 3}{3}\right] = \frac{n}{3}(4n^2 + 6n + 2 - 6n - 6 + 3)$

$= \frac{n}{3}(4n^2 - 1) = \frac{n}{3}(2n+1)(2n-1).$

MISCELLANEOUS EXERCISE ON CHAPTER 9

(Page No.: 199–201)

1. Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Sol. Let a be the first term and d, the common difference of A.P.

$$\begin{aligned} a_{m+n} + a_{m-n} &= [a + (m+n-1)d] + [a + (m-n-1)d] \\ &= 2a + (m+n-1+m-n-1)d \\ &= 2a + (2m-2)d \\ &= 2[a + (m-1)d] = 2a_m. \end{aligned}$$

$\therefore$ Sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms is equal to twice the m^{th} term.

2. If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Sol. Let the three numbers in A.P. be $a-d, a, a+d$.

Their sum = 24 $\Rightarrow (a-d) + a + (a+d) = 24$

$\Rightarrow \quad 3a = 24 \quad \therefore \quad a = 8$

Product of numbers = 440

$\Rightarrow (a-d)\,a\,(a+d) = 440 \Rightarrow (8-d) \times 8 \times (8+d) = 440$

$\Rightarrow 64 - d^2 = \frac{440}{8} = 55 \Rightarrow 64 - 55 = d^2 \Rightarrow d^2 = 9$

$\therefore \quad d = \pm 3.$

When $a = 8, d = 3,$ the numbers are

$8-3, 8, 8+3$ *i.e.,* 5, 8, 11.

When $a = 8, d = -3$ the numbers are

$8 - (-3), 8, 8 + (-3)$ *i.e.*, 11, 8, 5.

Hence, the numbers are 5, 8, 11 or 11, 8, 5.

3. Let the sum of n, $2n$, $3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$.

Sol. Let a be the first term and d, the common difference of A.P.

$$S_1 = \text{Sum of } n \text{ terms} = \frac{n}{2}[2a + (n-1)d]$$

$$S_2 = \text{Sum } 2n \text{ terms} = \frac{2n}{2}[2a + (2n-1)d]$$

$$S_3 = \text{Sum of } 3n \text{ terms} = \frac{3n}{2}[2a + (3n-1)d]$$

$$\therefore \text{R.H.S. } 3(S_2 - S_1) = 3\left[\frac{2n}{2}\{2a + (2n-1)d\} - \frac{n}{2}\{2a + (n-1)d\}\right]$$

$$= \frac{3n}{2}[2\{2a + (2n-1)d\} - \{2a + (n-1)d\}]$$

$$= \frac{3n}{2}[4a + (4n-2)d - 2a - (n-1)d]$$

$$= \frac{3n}{2}[4a - 2a + (4n - 2 - n + 1)d]$$

$$= \frac{3n}{2}[2a + (3n-1)d]$$

$$= S_3 = \text{L.H.S.}$$

Hence, $S_3 = 3(S_2 - S_1)$.

4. Find the sum of all numbers between 200 and 400 which are divisible by 7.

Sol.

7	200
	28 – 4

7	400
	57 – 1

Smallest multiple of 7 greater than 200 is $(200 - 4) + 7 = 203$.

Largest multiple of 7 less than 400 is $400 - 1 = 399$.

$\therefore$ Required sum $= 203 + 210 + 217 + ... + 399$.

It is an A.P. with $a = 203$, $d = 7$, $l = 399$.

Let $l = a_n = 399$, then $203 + (n-1) \times 7 = 399$.

$$\Rightarrow \quad (n-1) \times 7 = 399 - 203 = 196 \Rightarrow n - 1 = \frac{196}{7} = 28$$

$\Rightarrow \qquad n = 29$

$\therefore$ Required sum $= \frac{n}{2}(a + l) = \frac{29}{2}(203 + 399)$

$= \frac{29}{2} \times 602 = 29 \times 301 = 8729.$

5. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Sol. Let S_1 denote the sum of integers from 1 to 100 which are divisible by 2.

$\therefore S_1 = 2 + 4 + 6 + ... + 100$

$$\left(\text{an A.P. with } a = 2, l = 100, n = \frac{100}{2} = 50\right)$$

$= \frac{n}{2}(a + l) = \frac{50}{2}(2 + 100) = 25 \times 102 = 2550.$

Let S_2 denote the sum of integers from 1 to 100 which are divisible by 5.

$\therefore S_2 = 5 + 10 + 15 + ... + 100$

$$\left(\text{an A.P. with } a = 5, l = 100, n = \frac{100}{5} = 20\right)$$

$= \frac{n}{2}(a + l) = \frac{20}{2}(5 + 100) = 10 \times 105 = 1050.$

Now L.C.M. of 2 and 5 is 10. Multiples of 10 occur in S_1 as well as S_2. Let S_3 denote the sum of integers divisible by both 2 and 5 i.e. by 10.

$S_3 = 10 + 20 + 30 + ... + 100$

$$\left(\text{an A.P. with } a = 10, l = 100, n = \frac{100}{10} = 10\right)$$

$= \frac{n}{2}(a+l)$

$= \frac{10}{2}(10 + 100) = 5 \times 110 = 550$

$\therefore$ Required sum $= S_1 + S_2 - S_3$

$= 2550 + 1050 - 550 = 3600 - 550 = 3050.$

6. Find the sum of all two digit numbers which when divided by 4, yield 1 as remainder.

Sol. Two digit numbers are from 10 to 99.

Among them, multiples of 4 are 12, 16, 20, ..., 96.

$\therefore$ Two digit numbers which when divided by 4 yield 1 as remainder are 12 + 1 = 13, 17, 21, ..., 97 = (96 + 1).

Required sum = 13 + 17 + 21 + ... + 97.

It is an A.P. with $a = 13$, $d = 4$, $l = 97$.

Let $a_n = 97$, then $13 + (n - 1) \times 4 = 97$.

$\Rightarrow \quad (n - 1) \times 4 = 97 - 13 = 84 \Rightarrow \quad n - 1 = \frac{84}{4} = 21$

$\Rightarrow \quad n = 22$

$\therefore$ Required sum $= \frac{n}{2}(a + l) = \frac{22}{2}(13 + 97)$

$= 11 \times 110 = 1210.$

7. If f is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in$ N such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

Sol. $f(x + y) = f(x) f(y)$ ...(*i*) for all $x, y \in N$ (given)

$f(1) = 3$ ...(*ii*) (given)

Putting $x = 1, y = 1$ in (*i*), $f(2) = f(1) f(1) = 3 \times 3 = 9$ [By (*ii*)]

Putting $x = 1, y = 2$ in (*i*) $f(3) = f(1) f(2) = 3 \times 9 = 27$

and so on

Again $\sum_{x=1}^{n} f(x) = 120$ (given)

$\therefore \quad f(1) + f(2) + f(3) + ... + f(n) = 120$

Putting values of $f(1)$, $f(2)$, $f(3)$, ...

$3 + 9 + 27 + ...$ to n terms $= 120$

It is a G.P. with $a = 3$, $r = 3$

$\therefore \quad \frac{a(r^n - 1)}{r - 1} = 120$

Putting values of a and r.

$\frac{3(3^n - 1)}{3 - 1} = 120$ or $3(3^n - 1) = 240$

Dividing by 3, $3^n - 1 = 80$ or $3^n = 81 = 3^4$

$\therefore \quad n = 4$

8. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

Sol. Here $a = 5$, $r = 2 > 1$.

Let $S_n = 315$, then $\frac{5(2^n - 1)}{2-1} = 315$

$\Rightarrow 5(2^n - 1) = 315 \quad \Rightarrow 2^n - 1 = 63$

$\Rightarrow 2^n = 64 = 2^6 \quad \Rightarrow n = 6$

Last term $= a_6 = ar^5 = 5 \times 2^5 = 5 \times 32 = 160$.

9. The first term of a G.P. is 1. The sum of the third and fifth term is 90. Find the common ratio of G.P.

Sol. Let r be the C.R. of G.P. $\quad a = 1$

$a_3 = ar^2 = r^2 \qquad a_5 = ar^4 = r^4$

$a_3 + a_5 = 90 \text{ (given)} \quad \therefore \quad r^2 + r^4 = 90$

$\Rightarrow \quad r^4 + r^2 - 90 = 0$

Put $\quad r^2 = t$

$\therefore \quad t^2 + t - 90 = 0$

$D = b^2 - 4ac = 1 + 360 = 361$

$\therefore \; t = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm \sqrt{361}}{2}$

or $t = \frac{-1 \pm 19}{2} = 9, -10$

or $\quad r^2 = 9, -10 \qquad (\because \; t = r^2)$

But $r^2 = -10$ gives imaginary values of r and hence rejected.

$\therefore \quad r^2 = 9 \qquad \therefore \quad r = \pm 3$

10. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Sol. Let the three numbers in G.P. be a, ar, ar^2.

Their sum = 56 $\quad \Rightarrow \quad a + ar + ar^2 = 56$

$\Rightarrow \quad a(1 + r + r^2) = 56 \qquad \ldots(i)$

Subtracting 1, 7, 21 from the numbers, we get

$a - 1$, $ar - 7$, $ar^2 - 21$ which are given to be in A.P.

$\therefore \quad \text{II} - \text{I} = \text{III} - \text{II}$

$\therefore \quad (ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$

$\Rightarrow \quad ar - 7 - a + 1 = ar^2 - 21 - ar + 7$

$\Rightarrow \quad -a + 2ar - ar^2 \quad = \; -8$

$\Rightarrow \quad - a + 2ar - ar^2 \quad = \; -8$

$\Rightarrow \quad - a\,(1 - 2r + r^2) \; = -8$

Dividing by -1, $a\,(1 - 2r + r^2) = 8$ ...(*ii*)

Dividing (*i*) by (*ii*) to eliminate *a*,

$$\frac{1+r+r^2}{1-2r+r^2} = \frac{56}{8} = 7$$

Cross-multiplying, $7\,(1 - 2r + r^2) = 1 + r + r^2$

$\Rightarrow 7 - 14r + 7r^2 = 1 + r + r^2$

$\Rightarrow 6r^2 - 15r + 6 = 0$

Dividing by 3, $2r^2 - 5r + 2 = 0 \Rightarrow 2r^2 - 4r - r + 2 = 0$

$\Rightarrow 2r\,(r - 2) - (r - 2) = 0 \Rightarrow (r - 2)\,(2r - 1) = 0 \;\therefore\; r = 2, \frac{1}{2}.$

When $r = 2$, from (*i*), $a = 8$ and the numbers are a, ar, ar^2 *i.e.*, 8, 16, 32.

When $r = \frac{1}{2}$, from (*i*), $a = 32$ and the numbers are 32, 16, 8.

Remark: Here we have not taken three numbers in G.P. as $\frac{a}{r}$, a and ar because their product is not given.

11. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Sol. Let a be the first term and r, the common ratio of a G.P. consisting of $2n$ (even) terms.

Given: $a_1 + a_2 + a_3 + \ldots + a_{2n} = 5(a_1 + a_3 + a_5 + \ldots + a_{2n-1})$

$\Rightarrow \quad a + ar + ar^2 + \ldots + \text{to } 2n \text{ terms} = 5(a + ar^2 + ar^4 + \ldots + n \text{ terms})$

$$\Rightarrow \quad \frac{a(1-r^{2n})}{1-r} = 5 \times \frac{a(1-(r^2)^n)}{1-r^2}$$

Dividing by a, $\dfrac{1-r^{2n}}{1-r} = \dfrac{5(1-r^{2n})}{1-r^2} = \dfrac{5(1-r^{2n})}{(1-r)(1+r)}$

Dividing both sides by, $\dfrac{1-r^{2n}}{1-r}$, we have $1 = \dfrac{5}{1+r}$

$\Rightarrow \quad 1 + r = 5 \quad \therefore \quad r = 4.$

12. The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Sol. Here $a = 11$. Let d be the common difference of A.P.

Given: $S_4 = 56$

$\Rightarrow \frac{4}{2}[2 \times 11 + (4-1)d] = 56$

$\left[\because S_n \text{ of A.P} = \frac{n}{2}[2a + (n-1)d]\right]$

$\Rightarrow 2(22 + 3d) = 56 \Rightarrow 22 + 3d = 28$

$\Rightarrow 3d = 6 \quad \therefore d = 2$ ... (i)

Let l be the last term.

Sum of last four terms = 112 (given)

$\Rightarrow (l - 3d) + (l - 2d) + (l - d) + l = 112$

$\Rightarrow 4l - 6d = 112$

Putting $d = 2$ from (i), $4l - 12 = 112$

$\Rightarrow 4l = 124 \Rightarrow l = 31$

If n is the number of terms, then $a_n = l = 31$.

$\Rightarrow 11 + (n-1) \times 2 = 31 \Rightarrow (n-1) \times 2 = 20$

$\Rightarrow n - 1 = 10 \Rightarrow n = 11$

$\therefore$ Number of terms = 11.

13. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$, then show that a, b, c and d are in G.P.

Sol. From first two members,

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$$

Cross-multiplying, $(a + bx)(b - cx) = (a - bx)(b + cx)$

or $ab - acx + b^2x - bcx^2 = ab + acx - b^2x - bcx^2$

or $-acx + b^2x = acx - b^2x$

Transposing, $2b^2x = 2acx$

Dividing both sides by $2x$ $(x \neq 0$ (given)),

$b^2 = ac \Rightarrow b.b = a.c.$

$\therefore \frac{b}{a} = \frac{c}{b}$ ...(i)

Similarly from second and third members, we have

$$\frac{c}{b} = \frac{d}{c} \qquad ...(ii)$$

From (*i*) and (*ii*) $\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

$\therefore$ a, b, c, d, are in G.P.

14. Let S be the sum, P the product and R the sum of reciprocals of *n* terms in a G.P. Prove that $P^2R^n = S^n$.

Sol. $S = a + ar + ar^2 + ...$ to n terms

$$\therefore \quad S = \frac{a(1-r^n)}{1-r} \qquad ...(i)$$

$$P = a \,.\, ar \,.\, ar^2 \,...\, ar^{n-1}$$

$$= a^n . r^{1+2+...+(n-1)} = a^n . r^{\frac{n-1}{2}(1+n-1)}$$

$| \because$ In an A.P., $S_n = \frac{n}{2}(a + l)$

$$\therefore \quad P = a^n . r^{\frac{n(n-1)}{2}} \qquad ...(ii)$$

$$R = \text{Sum of reciprocals} = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + ... \text{ to } n \text{ terms}$$

It is a G.P. whose first term is $\frac{1}{a}$ and common ratio $\frac{1}{r}$.

Using S_n of G.P. $= a\frac{(1-r^n)}{1-r}$; we have

$$R = \frac{\frac{1}{a}\left[\left(\frac{1}{r}\right)^n - 1\right]}{\frac{1}{r}-1} = \frac{\frac{1}{a}\left(\frac{1}{r^n}-1\right)}{\frac{1}{r}-1} = \frac{\frac{1}{a}\left(\frac{1-r^n}{r^n}\right)}{\frac{1-r}{r}}$$

or $$R = \frac{1-r^n}{ar^n} . \frac{r}{1-r} = \frac{r(1-r^n)}{ar^n(1-r)} = \frac{1-r^n}{ar^{n-1}(1-r)} \qquad ...(iii)$$

L.H.S. = $P^2 R^n$

Putting values of P and R from (*ii*) and (*iii*)

$$\text{L.H.S.} = \left(a^n r^{\frac{n(n-1)}{2}}\right)^2 \left(\frac{(1-r^n)}{ar^{n-1}(1-r)}\right)^n$$

$$= a^{2n}\, r^{n(n-1)} \frac{(1-r^n)^n}{a^n r^{n(n-1)} (1-r)^n} = \frac{a^n(1-r^n)^n}{(1-r)^n} \quad ...(iv)$$

$$\text{Using } (i), \text{R.H.S. } S^n = \left(\frac{a(1-r^n)}{1-r}\right)^n = \frac{a^n(1-r^n)^n}{(1-r)^n} \quad ...(v)$$

From (*iv*) and (*v*), L.H.S. = R.H.S. *i.e.* $P^2 R^n = S^n$

15. The p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c, respectively. Show that

$$(q - r)a + (r - p)b + (p - q)c = 0.$$

Sol. Let A be the first term and d, the C.D. of A.P.

$$a_p = a \quad \Rightarrow \quad A + (p-1)d = a \quad ...(i)$$
$$a_q = b \quad \Rightarrow \quad A + (q-1)d = b \quad ...(ii)$$
$$a_r = c \quad \Rightarrow \quad A + (r-1)d = c \quad ...(iii)$$

Substituting the values of a, b, c from (*i*), (*ii*), (*iii*) L.H.S,

$= (q-r)\,a + (r-p)\,b + (p-q)\,c$

$= (q-r)\,(A + pd - d) + (r-p)\,(A + qd - d)$

$\quad + (p-q)\,(A + rd - d)$

$= qA + pqd - qd - Ar - prd + rd$

$= Ar + qrd - rd - Ap - pqd + pd$

$+\ Ap + prd - pd - qA - qrd + qd$

$= 0$ ($\because$ All terms cancel in pairs)

16. If $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$, $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

Sol. Given: $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$, $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$$\Rightarrow \frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab} \text{ are in A.P.}$$

Adding 1 to each term,

$$\frac{ab+ac}{bc} + 1, \frac{bc+ab}{ca} + 1, \frac{ca+bc}{ab} + 1 \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{ab+ac+bc}{bc}, \frac{bc+ab+ca}{ca}, \frac{ca+bc+ab}{ab} \text{ are in A.P.}$$

Dividing each term by $ab + bc + ca$,

$$\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab} \text{ are in A.P.}$$

Multiplying each term by abc,

a, b, c are in A.P.

17. If a, b, c, d are in G.P., prove that $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P.

Sol. Let r be the common ratio of G.P., then

$$b\ (= t_2) = ar,\ c = ar^2,\ d\ (= t_4) = ar^3$$

Now $a^n + b^n$, $b^n + c^n$, $c^n + d^n$ will be in G.P.

if $$\frac{b^n + c^n}{a^n + b^n} = \frac{c^n + d^n}{b^n + c^n} \qquad \left(\frac{\text{II}}{\text{I}} = \frac{\text{III}}{\text{II}}\right) \qquad ...(i)$$

L.H.S of (i) = $\dfrac{b^n + c^n}{a^n + b^n}$

Putting values of b and c, $= \dfrac{(ar)^n + (ar^2)^n}{a^n + (ar)^n}$

$$= \frac{a^n r^n + a^n r^{2n}}{a^n + a^n r^n} = \frac{a^n r^n (1 + r^n)}{a^n (1 + r^n)} = r^n$$

R.H.S of (i) = $\dfrac{c^n + d^n}{b^n + c^n}$

Putting values of b, c, d

$$= \frac{(ar^2)^n + (ar^3)^n}{(ar)^n + (ar^2)^n} = \frac{a^n r^{2n} + a^n r^{3n}}{a^n r^n + a^n r^{2n}}$$

$$= \frac{a^n r^{2n}(1 + r^n)}{a^n r^n (1 + r^n)} = \frac{r^{2n}}{r^n} = r^{2n-n} = r^n$$

$\therefore$ L.H.S of (i) = R.H.S of (i)

Hence (i) is true.

$\therefore$ $a^n + b^n$, $b^n + c^n$, $c^n + d^n$ are in G.P.

18. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P., prove that

$$(q + p) : (q - p) = 17 : 15.$$

Sol. Given: a, b are roots of $x^2 - 3x + p = 0$.

$$\Rightarrow \quad a + b = \frac{-B}{A} = \frac{-(-3)}{1} = 3 \quad \text{and} \quad ab = \frac{C}{A} = \frac{p}{1} = p$$

Also, c, d are roots of $x^2 - 12x + q = 0$.

$$\Rightarrow \quad c + d = \frac{-(-12)}{1} = 12 \quad \text{and} \quad cd = \frac{q}{1} = q.$$

Let r be the common ratio of G.P. a, b, c, d.

$\therefore \quad b = ar, c = ar^2, d = ar^3.$

$\therefore \quad a + b = 3 \quad \Rightarrow \quad a + ar = 3$

$\Rightarrow \quad a(1 + r) = 3$ ...(*i*)

and $\quad c + d = 12 \quad \Rightarrow \quad ar^2 + ar^3 = 12$

$\Rightarrow \quad ar^2(1 + r) = 12$ ...(*ii*)

Dividing (*ii*) by (*i*), (to eliminate a), $r^2 = 4$ ... (*iii*)

$$\text{L.H.S} = \frac{q+p}{q-p}$$

Putting values of $p = ab$ and $q = cd$,

$$= \frac{cd + ab}{cd - ab}$$

Putting $b = ar$, $c = ar^2$, $d = ar^3$;

$$= \frac{ar^2 ar^3 + a.ar}{ar^2 ar^3 - a.ar} = \frac{a^2r^5 + a^2r}{a^2r^5 - a^2r}$$

$$= \frac{a^2r\,(r^4+1)}{a^2r\,(r^4-1)} = \frac{r^4+1}{r^4-1} = \frac{(r^2)^2+1}{(r^2)^2-1}$$

Putting $r^2 = 4$, *from (iii)*, $= \dfrac{16+1}{16-1} = \dfrac{17}{15}$ = R.H.S

$\Rightarrow \quad q + p : q - p = 17 : 15.$

19. The ratio of the A.M. and G.M. of two positive numbers a and b, is $m : n$. Show that

$$a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2}).$$

Sol. Given: A.M. : G.M. = $m : n$

$$\Rightarrow \quad \frac{a+b}{2} : \sqrt{ab} = m : n \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$

By componendo and dividendo,
(Rule of Componendo and dividendo is:

$$\left.\text{if } \frac{a}{b} = \frac{c}{d}; \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d}\right)$$

$$\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{m+n}{m-n} \Rightarrow \frac{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 + 2\sqrt{a}\sqrt{b}}{\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{a}\sqrt{b}} = \frac{m+n}{m-n}$$

$$\Rightarrow \frac{(\sqrt{a} + \sqrt{b})^2}{(\sqrt{a} - \sqrt{b})^2} = \frac{m+n}{m-n}$$

Taking square root, $\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \dfrac{\sqrt{m+n}}{\sqrt{m-n}}$

Again by componendo and dividendo, we have

$$\frac{(\sqrt{a} + \sqrt{b}) + (\sqrt{a} - \sqrt{b})}{(\sqrt{a} + \sqrt{b}) - (\sqrt{a} - \sqrt{b})} = \frac{\sqrt{m+n} + \sqrt{m-n}}{\sqrt{m+n} - \sqrt{m-n}}$$

$$\Rightarrow \quad \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{m+n} + \sqrt{m-n}}{\sqrt{m+n} - \sqrt{m-n}}$$

$$\Rightarrow \quad \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{m+n} + \sqrt{m-n}}{\sqrt{m+n} - \sqrt{m-n}}$$

Squaring both sides,

$$\frac{a}{b} = \frac{(m+n) + (m-n) + 2\sqrt{(m+n)(m-n)}}{(m+n) + (m-n) - 2\sqrt{(m+n)(m-n)}}$$

$$= \frac{2m + 2\sqrt{m^2 - n^2}}{2m - 2\sqrt{m^2 - n^2}} = \frac{2\left(m + \sqrt{m^2 - n^2}\right)}{2\left(m - \sqrt{m^2 - n^2}\right)} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$$

$\Rightarrow \quad a : b = (m + \sqrt{m^2 - n^2}) : (m - \sqrt{m^2 - n^2})$.

20. If a, b, c are in A.P.; b, c, d are in G.P. and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that a, c, e, are in G.P.

Sol. We are given that

a, b, c are in A.P. Therefore, $b - a = c - b$ i.e. $2b = a + c$ ...(i)

b, c, d are in G.P. Therefore, $\frac{c}{b} = \frac{d}{c}$ i.e. $c^2 = bd$ ...(ii)

Because $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. (given)

therefore $\frac{1}{d} - \frac{1}{c} = \frac{1}{e} - \frac{1}{d}$

Transposing $\frac{2}{d} = \frac{1}{e} + \frac{1}{c} = \frac{c+e}{ce}$ $\therefore$ $d(c + e) = 2ce$

$\therefore \qquad d = \frac{2ce}{c+e}$ ...(iii)

To prove: a, c, e are in G.P.

Let us eliminate b and d as they are not required in the answer.

Putting the values of b and d from (i) and (iii) in (ii), we have

$$c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$$

or $$c^2 = \frac{(a+c)ce}{c+e}$$

Cross-multiplying and dividing by c,

$$c(c + e) = (a + c)e$$

or $$c^2 + ce = ae + ce$$

or $$c^2 = ae \Rightarrow \text{c.c.} = \text{ae} \Rightarrow \frac{c}{a} = \frac{e}{c}$$

$\therefore$ a, c, e are in G.P.

21. Find the sum of the following series up to n terms:

(i) 5 + 55 + 555 + ... (ii) .6 + .66 + .666 + ...

Sol. (i) $S_n = 5 + 55 + 555 + \ldots$ to n terms

$= 5(1 + 11 + 111 + \ldots$ to n terms$)$

$$= \frac{5}{9} \ (9 + 99 + 999 + \ldots \text{ to } n \text{ terms})$$

$$= \frac{5}{9} \ [(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{5}{9} \ [(10 + 10^2 + 10^3 + \ldots \text{ to } n \text{ terms})$$

$$- (1 + 1 + 1 + \ldots \text{ to } n \text{ times})]$$

[Using S_n of G.P. $= a\dfrac{(r^n - 1)}{r - 1}$ with $a = 10$ and $r = 10 > 1$]

$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$$

$$= \frac{50}{81}\ (10^n - 1) - \frac{5n}{9}.$$

(*ii*) $S_n = 0.6 + 0.66 + 0.666 + \ldots$ to n terms

Taking 6 common, $S_n = 6(0.1 + 0.11 + 0.111 + \ldots$ to n terms)

Divide and multiply by 9.

$$= \frac{6}{9}(0.9 + 0.99 + 0.999 + \ldots \text{ to } n \text{ terms})$$

$$= \frac{6}{9}[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + \ldots \text{ to } n \text{ terms}]$$

$$= \frac{6}{9}[(1 + 1 + 1 + \ldots n \text{ terms}) - (0.1 + 0.01 + 0.001$$

$$+ \ldots \text{ to } n \text{ terms})]$$

$$= \frac{6}{9}\left[n - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots \text{ to } n \text{ terms}\right)\right]$$

Using S_n of G.P. $= \dfrac{a(1 - r^n)}{1 - r}$ $\left(\text{with } a = \dfrac{1}{10}, r = \dfrac{1}{10} < 1\right)$,

$$= \frac{6}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10} = \frac{9}{10}}\right] = \frac{2}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{2}{3}\left[\frac{9n-1+\frac{1}{10^n}}{9}\right]$$

$$= \frac{2}{27}[9n-1+10^{-n}]$$

Remark: Don't take 0.6 common.

22. Find the 20th term of the series 2 × 4 + 4 × 6 + 6 × 8 + ... + *n* terms.

Sol. a_{20} = (20th term of A.P. 2, 4, 6, ...)(20th term of A.P. 4, 6, 8, ...)

$= [2 + (20 - 1) \times 2][4 + (20 - 1) \times 2]$

$= (2 + 38)(4 + 38) = 40 \times 42 = 1680.$

23. Find the sum of the first *n* terms of the series:

3 + 7 + 13 + 21 + 31 + ...

Sol. $S_n = 3 + 7 + 13 + 21 + 31 + ... + a_{n-1} + a_n$

Shifting every term one place to its right

$S_n = 0 + 3 + 7 + 13 + 21 ... + a_{n-1} + a_n$

On subtracting, we get

$0 = 3 + 4 + 6 + 8 + 10 + ...$ to n terms $- a_n$

Shifting $- a_n$ to L.H.S;

$\Rightarrow \quad a_n = 3 + [4 + 6 + 8 + 10 + ...$ to $(n - 1)$ terms$]$

$$= 3 + \frac{n-1}{2}[2 \times 4 + (n - 1 - 1) \times 2]$$

$$\left(\because \; S_n \text{ of A.P.} = \frac{n}{2}[2a + (n-1)\,d]\right)$$

$$= 3 + \frac{n-1}{2}(8 + 2n - 4) = 3 + \frac{n-1}{2}(2n + 4)$$

$= 3 + (n - 1)(n + 2)$

$= 3 + n^2 + n - 2 = n^2 + n + 1$

$\therefore \quad S_n = \Sigma n^2 + \Sigma n + 1.n$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= \frac{n}{6}[(n + 1)(2n + 1) + 3(n + 1) + 6]$$

$$= \frac{n}{6}(2n^2 + 3n + 1 + 3n + 3 + 6)$$

$$= \frac{n}{6}(2n^2 + 6n + 10) = \frac{n}{3}(n^2 + 3n + 5).$$

24. If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

Sol. According to given condition,

$$S_1 = \Sigma n = \frac{n(n+1)}{2}$$

$$S_2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = \Sigma n^3 = \frac{n^2(n+1)^2}{4}$$

$$\text{L.H.S.} = 9\,S_2^2 = 9\left(\frac{n(n+1)(2n+1)}{6}\right)^2 = 9\,\frac{n^2(n+1)^2(2n+1)^2}{36}$$

$$= \frac{n^2(n+1)^2(2n+1)^2}{4}$$

$$\therefore \text{R.H.S.} = S_3(1+8S_1) = \frac{n^2(n+1)^2}{4}\left[1+8.\frac{n(n+1)}{2}\right]$$

$$= \frac{n^2(n+1)^2}{4}(4n^2 + 4n + 1)$$

$$= \frac{n^2(n+1)^2}{4}(2n + 1)^2$$

$\therefore$ L.H.S. = R.HS *i.e.* $9\,S_2^2 = S_3(1+8S_1)$

25. Find the sum of the following series up to n terms:

$$\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots .$$

Sol. The n^{th} term of the given series is

$$a_n = \frac{1^3+2^3+3^3+\dots+n^3}{1+3+5+\dots \text{ to } n \text{ terms}}$$

For denominator, using S_n of A.P. $= \frac{n}{2}(2a + (n-1)d)$

$$a_n = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}[2.1+(n-1).2]} = \frac{n^2(n+1)^2}{4.\frac{n}{2}.(2+2n-2)} = \frac{n^2(n+1)^2}{4.\frac{n}{2}.2n}$$

$$\Rightarrow a_n = \frac{(n+1)^2}{4} = \frac{n^2+2n+1}{4} = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}$$

$$\therefore \quad S_n = \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}.n$$

$$= \frac{1}{4} . \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} . \frac{n(n+1)}{2} + \frac{n}{4}$$

$$= \frac{n}{4}\left[\frac{(n+1)(2n+1)}{6} + (n+1) + 1\right]$$

$$= \frac{n}{4}\left[\frac{(2n^2+3n+1)+6(n+1)+6}{6}\right]$$

$$= \frac{n}{24}(2n^2 + 9n + 13).$$

26. Show that $\dfrac{1\times 2^2 + 2\times 3^2 + ... + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + ... + n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$.

Sol. For the numerator, $a_n = n(n+1)^2 = n(n^2 + 2n + 1)$

$$= n^3 + 2n^2 + n$$

$$\therefore \text{Numerator} = \Sigma n^3 + \Sigma n^2 + \Sigma n$$

$$= \frac{n^2(n+1)^2}{4} + 2.\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{(2n+1)}{3} + \frac{1}{2}\right]$$

$$= n(n+1)\frac{[3n(n+1)+4(2n+1)+6]}{12}$$

$$= \frac{n(n+1)(3n^2+11n+10)}{12}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \qquad ...(i)$$

$[\because \ 3n^2 + 11n + 10 = 3n^2 + 6n + 5n + 10$
$= 3n(n+2) + 5(n+2) = (n+2)(3n+5)]$

For the denominator, $a_n = n^2(n + 1) = n^3 + n^2$ on expanding

$\therefore$ Denominator $= \sum n^3 + \sum n^2$

$$= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= n(n+1)\left[\frac{n(n+1)}{4} + \frac{(2n+1)}{6}\right]$$

$$= n(n+1)\frac{[3n(n+1)+2(2n+1)]}{12}$$

$$= \frac{n(n+1)(3n^2+7n+2)}{12}$$

$$= \frac{n(n+1)(n+2)(3n+1)}{12} \qquad ...(ii)$$

[$\because 3n^2 + 7n + 2 = 3n^2 + 6n + n + 2 = 3n(n + 2)$
$+ 1(n + 2) = (n + 2)(3n + 1)$]

Dividing (*i*) by (*ii*), we get

$$\frac{1\times 2^2 + 2\times 3^2 + ... + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + ... + n^2\times(n+1)} = \frac{3n+5}{3n+1}.$$

27. A farmer buys a used tractor for ₹ 12,000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalments of ₹ 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

Sol. Cost of tractor = ₹ 12,000

Cash paid = ₹ 6,000

$\therefore$ Balance = ₹ 6,000

Number of instalments $= \frac{6000}{500} = 12$

[$\because$ Each instalment = ₹ 500 (given)]

Interest paid with 1st instalment

= Interest on unpaid amount ₹ 6000 for 1 year

$= ₹\ \frac{6000\times 12\times 1}{100} = ₹\ 720$ $\quad \left| \frac{P\times R\times T}{100} \right.$

Interest paid with 2nd instalment

= Interest on unpaid amount ₹ 5500

(= 6000–500) for 1 year)

$$= ₹\ \frac{5500 \times 12 \times 1}{100} = ₹\ 660$$

Interest paid with 3rd instalment

= Interest on unpaid amount ₹ 5000 (= 5500–500) for 1 year

$$= ₹\ \frac{5000 \times 12 \times 1}{100} = ₹\ 600$$

and so on

∴ Total interest paid

= ₹ (720 + 660 + 600 + ... + to 12 terms)

[an A.P. with a = 720, d = –60, n = 12]

$$= \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{12}{2}\left[2(720)+(12-1)(-60)\right] = 6(1440-660)$$

$$= 6\ (780) = ₹\ 4680$$

Total cost of tractor = ₹ 12,000 + ₹ 4,680 = ₹ 16680.

28. Shamshad Ali buys a scooter for ₹ 22,000. He pays ₹ 4000 cash and agrees to pay the balance in annual instalment of ₹ 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Sol. Total cost of the scooter = ₹ 22000

Cash paid = ₹ 4000

∴ Balance = ₹ 18000

Number of instalments @ ₹ 1000 each $= \frac{18000}{1000} = 18$

Interest on first instalment $= ₹\ \frac{18000 \times 10 \times 1}{100}$

= ₹ 1800

∴ According to given first instalment = ₹ (1000 + 1800)

= ₹ 2800

Interest on second instalment $= ₹\ \frac{17000 \times 10 \times 1}{100}$

[∴ Balance after first instalment = ₹ (18000 – 1000 = 17000)]

$\therefore$ Second instalment = ₹ (1000 + 1700)
= ₹ 2700

Interest on third instalment = ₹ $\frac{16000 \times 10 \times 1}{100}$
= ₹ 1600

$\therefore$ Third instalment = ₹ (1000 + 1600)
= ₹ 2600 and so on.

$\therefore$ Total amount paid in instalments = ₹ (2800 + 2700 + 2600 + ... to 18 terms)

[It is an A.P. here $a = 2800, d = -100, n = 18$]

$$= \frac{n}{2}[2a + (n-1)d]$$

$$= ₹\frac{18}{2}[2(2800) + (18-1)(-100)]$$

= ₹ 9 (5600–1700) = ₹ 9 (3900) = ₹ 35100

$\therefore$ Total cost of scooter = ₹ (4000 + 35100) = ₹ 39100

Remark: Q.N. 27 can also be done by the method of Q.N. 28 and Q.N. 28 can be done by the method of Q.N. 27.

29. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letters is mailed.

Sol. Total number of letters in 8 sets = 4 + (4 × 4) + (16 × 4) ... to 8 terms

= 4 + 16 + 64 + ... to 8 terms

(a G.P. with $a = r = 4 > 1, n = 8$)

$$= a\frac{(r^n - 1)}{r-1} = \frac{4(4^8 - 1)}{4-1} = \frac{4}{3}(65536 - 1) = \frac{4}{3} \times 65535$$

= 4 × 21845 = 87380

$\therefore$ Total amount spent on postage

$$= ₹\left(\frac{50}{100} \times 87380\right) = ₹\ 43690.$$

30. A man deposited ₹ 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

Sol. Principal = ₹ 10000

Amount at the end of 1 year

$$= ₹\left(10000+\frac{10000\times 5\times 1}{100}\right) = ₹\ (10000 + 500)$$

= ₹ 10500

Amount at the end of 2 years

$$= ₹\left(10000+\frac{10000\times 5\times 2}{100}\right) = ₹\ (10000 + 1000)$$

= ₹ 11000

Amount at the end of 3 years

$$= ₹\left(10000+\frac{10000\times 5\times 3}{100}\right) = ₹\ (10000 + 1500)$$

$$= ₹\ 11500 \qquad \left(\because S.I = \frac{P\times R\times T}{100}\right)$$

and so on

∴ The amounts at the end of the 1st, 2nd, 3rd, ... years are ₹ 10500, ₹ 11000, ₹ 11500, ...

They form an A.P. with a = 10500, d = 500

∴ Amount in 15th year

= Amount at the end of 14 years

$= a_{14} = a + 13d$

= ₹ (10500 + 13 × 500) = ₹ (10500 + 6500)

= ₹ 17,000

Amount after 20 years = $a_{20} = a + 19d$

= ₹ (10500 + 19 × 500)

= ₹ (10500 + 9500)

= ₹ 20000.

31. A manufacturer reckons that the value of a machine, which costs him ₹ 15625, will depreciate

each year by 20%. Find the estimated value at the end of 5 years.

Sol. Rate of depreciation = 20% each year

Initial cost V_0 = ₹ 15625.

Depreciated value at the end of 1st year

$$= V_0 - \frac{20}{100} V_0 = V_0 \left(1 - \frac{1}{5}\right)$$

$$= V_0 \left(\frac{4}{5}\right) = V_1$$

Depreciated value at the end of 2nd year

$$= V_1 - \frac{20}{100} V_1 = V_1 \left(1 - \frac{1}{5}\right)$$

$$= V_1 \left(\frac{4}{5}\right) = V_0 \left(\frac{4}{5}\right)\left(\frac{4}{5}\right) = V_0 \left(\frac{4}{5}\right)^2$$

and so on.

Depreciated value at the end of n^{th} year

$$= V_0 \left(\frac{4}{5}\right)^n$$

∴ Depreciated value at the end of 5 years

$$= V_0 \left(\frac{4}{5}\right)^5$$

Putting the value of V_0 = ₹ 15625,

$$= ₹\ 15625 \times \frac{1024}{3125} = ₹\ (5 \times 1024)$$

$$= ₹\ 5120.$$

32. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Sol. 150 workers were engaged to finish a job in a certain number of days say k days.

∴ Number of workers who would have worked for k days

$$= (150 + 150 + 150 + \ldots\ldots + k \text{ terms})$$

$$= 150k \qquad \ldots(i)$$

But workers present on first day are 150 (given), on second day 150 – 4 = 146

($\because$ 4 workers dropped on second day),

on third day = 146 – 4 = 142 and so on.

Because of the dropping out of workers, it took 8 more days to finish the work (given)

$\therefore$ Number of days taken to finish the work

$$= k + 8 = n \text{ (say)} \quad \ldots(ii)$$

$\therefore$ 150 + 146 + 142 + ... to n terms (days) = 150 k [By (i)]

The left hand series is an A.P. with a = 150,

$$d = 146 - 150 = -4,$$

$$\Rightarrow \quad \frac{n}{2}[2a + (n-1)d] = 150k$$

Putting $k = n - 8$ from (ii),

$$\Rightarrow \quad \frac{n}{2}[300 + (n-1)(-4)] = 150(n-8)$$

$$\Rightarrow \quad \frac{n}{2}[300 - 4n + 4] = 150(n-8)$$

$$\Rightarrow \quad n(304 - 4n) = 300(n-8)$$

$$\Rightarrow \quad 304n - 4n^2 = 300n - 2400$$

$$\Rightarrow \quad -4n^2 + 4n + 2400 = 0$$

Dividing by – 4, $\quad n^2 - n - 600 = 0$

$$\Rightarrow \quad n^2 - 25n + 24n - 600 = 0$$

$$\Rightarrow \quad n(n-25) + 24(n-25) = 0$$

$$\Rightarrow \quad (n-25)(n+24) = 0$$

$\Rightarrow$ Either $n - 25 = 0$ or $n + 24 = 0$

i.e., $n = 25$ or $n = -24$

But $n = -24$ is impossible because number of days can't be negative.

$\therefore \quad n = 25.$